KINGDOM OF SAUDI ARABIA
MINISTRY OF HIGHER EDUCATION
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COLLEGE OF ENGINEERING

# INTRODUCTION TO ENGINEERING FLUID MECHANICS

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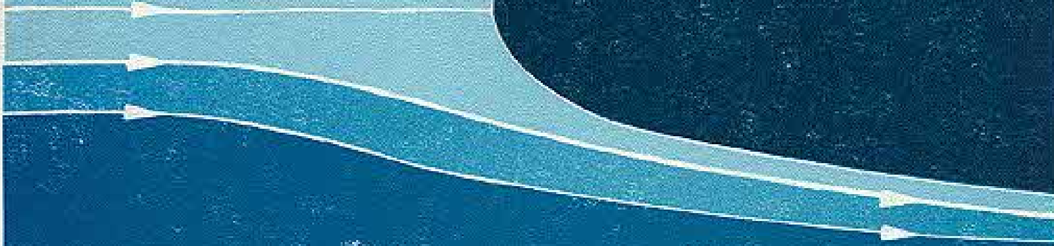

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بسم الله الرحمن الرحيم

IN THE NAME OF ALLAH,
THE MERCIFUL
THE MERCY-GIVING

## ACKNOWLEDGEMENT

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## FORWARD

This book is intended to familiarise students with the analytical approach to the essential fundamentals of fluid mechanics. Hence it dwelled on simple but rigorous applications of the physics laws to continuum and channel flows of fluids. Ideal flow conditions were assumed in certain parts of the analysis. In other parts the concept of friction (viscosity) was introduced to present the flow of real fluids to the students. Elaborate solved examples were given where it was felt necessary.

It is believed that in this form- the book should offer fairly broad foundations of fluid mechanics for engineering students presented in simple analytical but concise manner. It is hoped that engineering students when presented by this introduction will find it easy to handle further studies of fluid mechanics whatever their specializations may be, civil, mechanical aeronautical or chemical.

The book may be used as a text for students taking the first course in fluid mechanics. Three hours lecturing per week for one semester should be enough for such course if supplemented by reasonable tutorial periods.

## CONTENTS

## CHAPTER ONE
## FUNDAMENTALS

### 1 – 1 – Fluids

A fluid is a material whose particles would be displaced under the action of the slightest shearing stress. A material possessing this property will flow readily under the effect of any magnitude of shear stresses. If the rate of shear deformation is small, the fluid offers negligible resistance. With increased rate of deformation it offers increased resistance. The fluid resistance disappears, however, once the deforming motion ceases. The resistance arises from the existence of viscosity in the fluid.

Fluids are conventionally classified as

a – Liquids, or

b – Gases and Vapours.

A *liquid* is a fluid which, at a given temperature occupies a definite volume which is little affected by change in pressure. Poured into a stationary container, the liquid will occupy the lower part of the container and form a free level surface. Liquids offer considerable resistance (comparable to that of solids) to compression stresses, but they offer negligible resistance to tensile stresses.

A *gas or vapour* has the property that when a quantity of it, however small, is placed in an otherwise empty closed vessel it fills the vessel completely. In other words gases and vapours would expand unless constrained, and because of this property it seems impossible to create tensile stresses in a gas. Gases, unlike liquids, offer widely varying resistance to compression stresses, the variation depending on different parameters.

The properties which can be used to distinguished between liquids and gases are the following

1. The bulk modulus of elasticity which indicates the ability of the fluid for compression. Gases are normally easier to be compressed than liquids.
2. The density of the fluid: where gases are normally less dense than liquids.

The ideal fluid is an imaginary fluid which does not have any internal friction and may be called non-viscous fluid or inviscid fluid.

## 1 – 2 – Continuum and Fluid Properties

Fluids are made of molecules and between the molecules are voids (much longer than the molecules themselves). The molecules are in continuous random motion. In dealing with fluid mechanics one becomes interested in the overall motion of the fluid instead of the motion of each molecule. The continuum concept is to assume that the fluid is a continuous medium instead of molecules with mean free paths between the molecules. This concept of continuum is widely accepted by engineers and is found to help establishing an accurate rule for the determination of average properties for various fluids.

For example, the density of a fluid continuum at a point is defined as

$$\rho = \lim_{V \to V^*} \frac{m}{V} \tag{1.1}$$

where m is the mass of a fluid contained in the volume V at the limit when the volume V is reduced to its lowest possible limit $V^*$. The limiting volume $V^*$ although is very small it still should be consistent with the continuum concept, i.e. it contains a large number of molecules. To get an idea about how small is $V^*$ it may become worthy to mention that 1 $cm^3$ of air at normal temperature and pressure contains about $2 \times 10^{19}$ molecules.

Similarly the pressure of the fluid at a point is defined as follows

$$p = \lim_{A \to A^*} \frac{F}{A} \tag{1.2}$$

where F is the time-average normal force exerted by the molecules on the surface A as A tends to its lower limit of $A^*$. Again the dimension of $A^*$ must be much larger than the distance between the molecules.

The specific volume of a fluid v is defined as follows

$$v = \frac{1}{\rho} \tag{1.3}$$

while the specific gravity s of a fluid is a measure of comparing the fluid density to the density of water at a given reference temperature, thus

$$s = \frac{\rho}{\rho_\omega} \tag{1.4}$$

The specific weight $\gamma$ of a fluid is defined as follows

$$\gamma = \rho g \tag{1.5}$$

Table (F.2), (F.3), and (F.4) give the values of ρ for various fluids.

## 1 – 3 – The Perfect Gas

A perfect gas is defined as the gas that obeys the relation

$$pv = RT \tag{1.6}$$

where p is the absolute pressure of the gas, v is its specific volume, R is the gas constant and T is its absolute temperature. Other variants of the perfect gas relation are given as follows

$$p = \rho R T \tag{1.7}$$

$$pV = m R T \tag{1.8}$$

where ρ is the density of the gas V is its volume and m is its mass. Values of the gas constant for various gases are given in Table (F.5).

## 1 – 4 – Process

The path of all states of the fluid as it changes from one state to another is known by a "process". A fluid can go from an initial state to a final state via many possible processes. The most important processes are defined below.

An isothermal process is the process which keeps the temperature constant. For a perfect gas, using Eq (1.6) the isothermal process is then described by the following equation

$$p\,v = \text{const} \tag{1.9}$$

The isobaric process is the process that maintains constant pressure. If the process is carried out such that no heat is exchanged between the fluid and the surroundings then the process is said to be "adiabatic". When the process is also "reversible" (i.e. frictionless) in addition to being adiabatic the process is then called "isentropic". For a perfect gas the isentropic process is described by the relation

$$pv^{\gamma} = \text{constant} \tag{1.10}$$

where $\gamma$ is the ratio of the specific heat at constant pressure to that at a constant volume. The value of $\gamma$ for various gases is given in Table (F.5).

The following relations can be derived from Eqs. (1.6) and (1.10) for a perfect gas under an isentropic process

$$\frac{T_2}{T_1} = \left( \frac{p_2}{p_1} \right)^{(\gamma-1)/\gamma} \tag{1.11}$$

$$\frac{\rho_2}{\rho_1} = \left( \frac{p_2}{p_1} \right)^{1/\gamma} \tag{1.12}$$

**EXAMPLE 1 – 1**

Air at 20°C and 100 k Pa abs is compressed to 200 k Pa abs. Calculate the final temperature and density if the compression process is

a) isothermal    b) isentropic

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Data of the Problem

* Initial conditions of air:
  $T_1 = 20 + 273 = 293$ K
  $P_1 = 100$ kPa abs $= 10^5$ Pa abs
* Final pressure $= P_2 = 2 \times 10^5$ Pa abs

Requirements

* $T_2$ and $\rho_2$ for
  a) isothermal compression
  b) isentropic compression

Solution

a) Isothermal compression

$T_2 = T_1 = 20°$ C

From Eq. (1.7) we have

$$\rho_2 = \frac{p_2}{RT_2}$$

From Appendix F, R = 287 J/kg K, then

$$\rho_2 = \frac{2x10^5}{287x293} = 2.38 \quad kg/m^3$$

b) Isentropic compression

Using Eq. (1.11),

$$T_2 = T_1 \left( \frac{p_2}{p_1} \right)^{(\gamma-1)/\gamma}$$

The value of $\gamma$ is obtained from Table (F.5) as 1.4, then

$$T_2 = 293 \left( \frac{2}{1} \right)^{\frac{0.4}{1.4}} = 357.17 \quad K$$

$$= 84.17^{\circ}C$$

Using Eq. (1.7)

$$\rho_2 = \frac{p_2}{RT_2} = \frac{2x10^5}{287x357.17} = 1.95 \quad kg/m^3$$

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## 1 – 5 – Vapour Pressure

The boiling temperature of a liquid depends on the acting pressure on the liquid. This boiling temperature increases with the increase of the acting pressure and vice versa. For example water boils at about 99.64°C when the pressure is $10^5$ Pa, at 81.35°C if $p = 0.5 \times 10^5$ Pa, and at 32.88°C if $P = 0.05 \times 10^5$ Pa. The pressure required for boiling of a liquid at a given temperature is termed the "vapour pressure". At a given temperature a liquid will boil if the acting pressure on the fluid is equal to or less than the vapour pressure of the liquid at the given temperature. Table (F. 6) illustrates the vapour pressures for various liquids at different temperatures.

The importance of the vapour pressure may be well understood by taking the example of water flowing in a pipe system. At normal ambient temperature water may boil in some parts of the pipe system if the local pressure becomes lower than the vapour pressure of the water corresponding to the operating water temperature. This phenomenon is known as cavitation and may cause a serious reduction of the flow rate, due to evaporation of water, and consequently may burn the pumps. Vapour pressure is also important in many other applications in fluid mechanics.

**EXAMPLE 1 – 2**

Which of the following fluids may boil?

a) water at 20°C and 2 kPa abs

b) water at 40°C and 2 kPa abs

c) mercury at 15°C and 0.1 kPa abs.

d) benzene at 20°C and 9 kPa abs.

---

Data of the Problem

a) water: T = 20°C , P = 2 kPa abs

b) water: T = 40°C , P = 2 kPa abs

c) mercury: T = 15°C , P = 0.1 kPa abs

d) benzene: T = 20°C , P = 9 kPa abs

Requirements

Check whether the fluid will boil or not.

Solution

Using Table (F. 6) we get the following

a) water at 20°C, $(P_V)_{20}$ = 2.34 kPa abs

i.e. water boils since $P < P_V$

b) water at 40°C, $(P_V)_{40} > (P_V)_{20}$

since $P < (P_V)_{20}$

then P must be less than $(P_V)_{40}$

i.e. water boils

c) mercury at 15°C, $(P_V)_{15.6}$ = 0.00017 kPa abs

but $(P_V)_{15} < (P_V)_{15.6}$
Therefore, $P > (P_V)_{15.6} > (P_V)_{15}$
i.e. mercury does not boil

d) benzene at 20°C , $P_V$ = 10 kPa abs
since $P < P_V$
Then benzene boils

---

## 1 – 6 – Compressibility

Fluids can be deformed by two means: shearing or compression. When a fluid is compressed, the degree of compression depends on the type of fluid, the original volume, and the applied pressure. Theoretically, a fluid may be treated as an elastic medium that stores the compression energy and recovers it as the applied pressure is removed. Similar to solids, the fluid modulus of elasticity, or the fluid bulk modulus is defined as follows

$$K = - \frac{dp}{dV/V} \tag{1.13}$$

where dp is the change of pressure and dV is the change in the original volume V. Other variants of Eq. (1.13) can be derived as below

$$K = - \frac{dp}{dv/v} \tag{1.14}$$

$$K = \frac{dp}{d\rho/\rho} \tag{1.15}$$

The bulk modulus K is a property of the fluid and its value depends on the acting pressure and the fluid temperature. Also, the value of K depends on the type of the compression process. Thus for isothermal compression, the isothermal bulk modulus of the fluid becomes

$$K_{isoth} = - \left(\frac{dp}{dV/V}\right)_T = - \left(\frac{dp}{dv/v}\right)_T = \left(\frac{dp}{d\rho/\rho}\right)_T \tag{1.16}$$

Another property which is of importance in fluid mechanics is the coefficient of compressibility defined as follows

$$\beta = \frac{1}{K} = - \frac{1}{V} \frac{dV}{dp} \tag{1.17}$$

$$\beta = \frac{1}{\rho} \frac{d\rho}{dp} \tag{1.18}$$

Generally, liquids are harder to compress than gases. For this reason it is common under most cases to treat liquids as incompressible fluids. Typical values of the isothermal bulk modulus K of water at 20°C and atmospheric pressure is $2.18 \times 10^9$

Pa while that of air at the same conditions of temperature and pressure is $10^5$ Pa.

For perfect gases the expressions for K can be derived through using Eqs. (1.7), (1.10), and (1.15). The results for isothermal and isentropic compression are, respectively, as follows

$$K_{isoth} = p \tag{1.19}$$

$$K_{isent} = \gamma p \tag{1.20}$$

**1–7– Speed of Sound**

The speed of sound in a fluid is defined in terms of the isentropic bulk modulus $K_{isent}$ as follows

$$a = \sqrt{\frac{1}{\rho} \; K_{isent}} \tag{1.21}$$

The speed of sound in a perfect gas under an isentopic process then becomes

$$a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma RT} \tag{1.22}$$

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**EXAMPLE 1–3**

Calculate the speed of sound in the following fluids

a) air at 20°C and 100 kPa abs

b) air at 100°C and 100 kPa abs

c) water at 20°C and 100 kPa abs

Data of the Problem

a) air: T = 20°C , p = $10^5$ Pa abs

b) air: T = 100°C , p = $10^5$ Pa abs

c) water: T = 20°C , p = $10^5$ Pa abs

Requirements

To calculate the speed of sound in each case

Solution

a) From Table (F.5), $\gamma$ = 1.4, R = 287.1. Using Eq. (1.22) we get

$$a = \sqrt{\gamma RT} = \sqrt{1.4 x 287.1 x (20+273)} = 343.2 \quad m/s$$

b) Assuming negligible change in the value of $\gamma$ with temperature, then

$$a = \sqrt{\gamma RT} = \sqrt{1.4 x 287.1 x (100+273)} = 387.2 \quad m/s$$

c) From Table (F.6), K = 2.069 × $10^9$ Pa, then using Eq. (1.21) gives

$$a = \sqrt{\frac{K}{\rho}} = \sqrt{\frac{2.069 \times 10^9}{1000}} = 1438.4 \quad m/s$$

## 1 – 8 – Viscosity

Viscosity is the property which makes fluids offer resistance to shearing. Shearing in fluids is manifested when adjacent particles travel at different velocities. Therefore, in a uniform constant velocity flow shearing would not exist.

It is established by many and thorough investigations particularly of the flow of liquids in pipes, that if a fluid flows over a rigid boundary there will be no relative

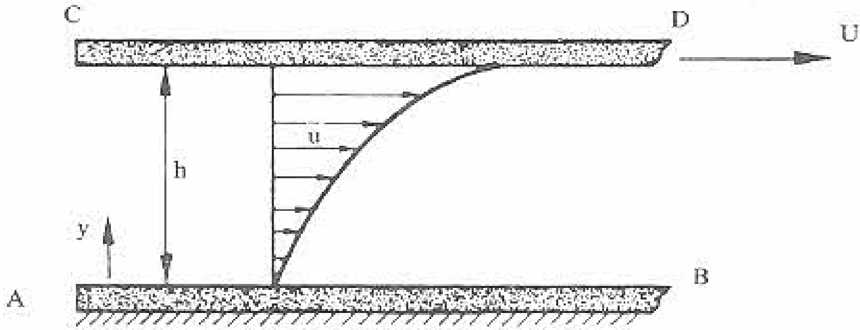


*Fig. 1.1 Fluid motion between two parallel flat plates*

velocity at the common surface between the fluid and the boundary (this is known as the no slip condition). Having this fact in mind, let us imagine two parallel flat plates AB and CD, Fig. (1.1), at a distance h from each other, with the space between them filled with a uniform fluid and the upper plane CD moving uniformly and unidirectionally with velocity U. According to the no slip condition it now follows that the layer of the fluid adjacent to the plate AB will be stand still, while the layer adjacent to the plate CD will be travelling at a velocity U. The layers of the fluid filling the space between the two plates will be travelling at different velocities u ranging from almost zero near to the plane AB and increasing gradually to a velocity almost equal to U near the plate CD. The observation showed that for a certain class of fluids the shear stress at any layer at distance y from plate AB vary linearly with the velocity gradient with respect to the distance y, i.e.

$$\tau \propto \frac{du}{dy}$$

or

$$\tau = \mu \frac{du}{dy} \qquad (1.23)$$

where $\mu$ is called the dynamic viscosity or simply the viscosity. The units of the dynamic viscosity is Pa s, i.e. $N.s/m^2$.

Real fluids have non-zero values for their viscosities. An ideal fluid is thus defined as the fluid with zero viscosity and may also be called non-viscous fluid. The value of the dynamic viscosity of a fluid is generally temperature dependent. For liquids the value of $\mu$ decreases with the increase in the temperature while for gases, $\mu$ increases with the increase in the temperature. Values of $\mu$ for various liquids and gases are given in Fig. (F. 1) for various temperatures.

The kinematic viscosity of the fluid $\nu$ is defined as the ratio of the dynamic viscosity of the fluid to its density, i.e.

$$\nu = \frac{\mu}{\rho} \tag{1.24}$$

The importance of this quantity appears in many fluid mechanics applications where the ratio $\mu/\rho$ appears quite often. Values of $\nu$ for various liquids and gases are given in Figs. (F. 2) and (F. 3), respectively.

The fluids that follow Eq. (1.23) with $\mu$ independent of the velocity gradient du/dy are called Newtonian fluids. Examples of Newtonian fluids are water, air, oil, kerosene, helium and many others. Non-Newtonian fluids are those fluids that do not follow Eq. (1.23) unless $\mu$ depends on the velocity gradient du/dy. Example of non-Newtonian fluids are blood, tar and slurries.

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**EXAMPLE 1 – 4**

In Fig. (1.1) if U = 20 m/s and h = 2 mm, calculate the shear stress on each plate when water is the fluid between the two plates.

Data of the Problem

* U = 20 m/s
* h = 2 mm
* Fluid is water
* Linear variation in the velocity

Requirements

* shear stress on the upper and lower plates

Solution

For linear variation of velocity,

$$\frac{du}{dy} = \frac{U - o}{h} = \frac{U}{h}$$

Assuming room temperature and using Fig (F1. 1) gives
$\mu$ = 0.001 Pa.s. Using Eq. (1.23) we get

$$(\tau)_{upper} = \mu\left(\frac{du}{dy}\right)_{y=h} = 0.001 \times \frac{20}{0.002} = 10 \text{ Pa}$$

$$(\tau)_{lower} = \mu\left(\frac{du}{dy}\right)_{y=o} = 0.001 \times \frac{20}{0.002} = 10 \text{ Pa}$$

---

## 1 – 9 – Surface Tension

Surface tension is a property of a fluid caused by the forces of attraction between like molecules called cohesion and those of unlike molecules called adhesion. At the interior of a fluid the cohesive forces balance each other. However at the separating layer between the fluid and another fluid or solid the adhesive and cohesive forces are not balanced. The net force of the cohesive and adhesive forces is known as the surface tension force $F_{st}$ which is proportional to the length of the free surface $L_{fs}$. Thus, one may write that.

$$F_{st} \alpha L_{fs}$$

or

$$F_{st} = \sigma L_{fs} \qquad (1.25)$$

where $\sigma$ is the coefficient of surface tension which is a property of the two fluids around the free surface. Table (F. 7) shows values of $\sigma$ for various fluid combinations.

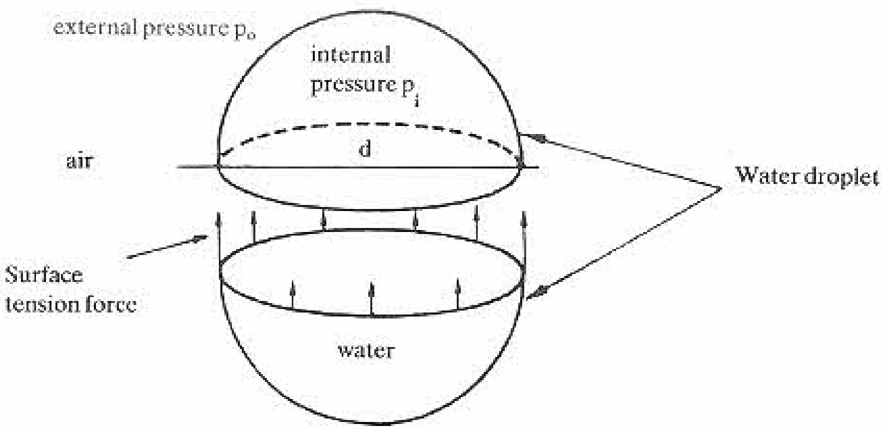


*Fig. 1.2 Balance of water droplet under surface tension and external and internal pressures*

One of the examples of the applications of the surface tension is water droplets which takes its shape and keeps the water molecules together as a result of the surface tension. Assuming a spherical water drop as that shown in Fig (1.2), with internal pressure $P_i$ and external surrounding pressure $P_o$ the application of Eq. (1.25) gives

$$F_{st} = (p_i - p_o) \frac{\pi}{4} d^2 = \sigma . \pi d$$

or

$$p_i - p_o = \frac{4\sigma}{d} \qquad (1.26)$$

where d is the diameter of the water droplet.

Another application of the surface tension is the capillary action which is illustrated as follows. Consider, for example, a vertical capillary tube (of small diameter) partially immersed in a liquid as illustrated in Fig (1.3a). The liquid inside the tube may rise or depress depending on the wetability of the liquid to the solid surface. Liquids that wet the tube walls rise and those do not wet the walls depress as illustrated by Figs (1.3b and c), respectivelly. The forces balance of the liquid rise of Fig (1.3a) gives

$$F_{st} \cos\theta = \text{weight of liquid rise}$$

i.e.

$$\sigma\pi d \cos\theta = \rho g \left( \frac{\pi}{4} d^2 \right) h$$

or,

$$h = \frac{4\sigma \cos\theta}{\rho g d} \qquad (1.27)$$

where $\theta$ is the angle of contact between the liquid free surface and the solid wall. The above equation indicates that the liquid rises, i.e., positive h, if the liquid wets the surface ($\theta < 90$) and the liquid depresses, i.e. negative h, if the liquid does not wet the surface ($\theta > 90$).

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**EXAMPLE 1−5**

Calculate the difference between the inside and outside pressure of a water droplet of diameter 2mm.

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Data of the Problem

* water droplet, d = 0.002 m

Requirements

* Calculate $P_i - P_o$

Solution

Using Table (F.7) gives, $\sigma = 0.073$ N/m

Applying Eq. (1.26) yields,

$$p_i - p_o = \frac{4\sigma}{d}$$

$$= \frac{4 \times 0.073}{0.002} = 146 \text{ Pa}$$

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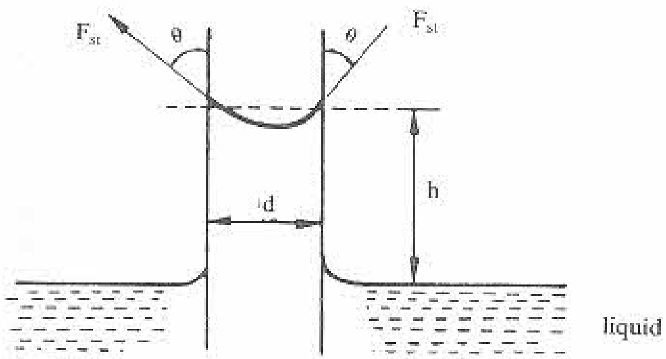
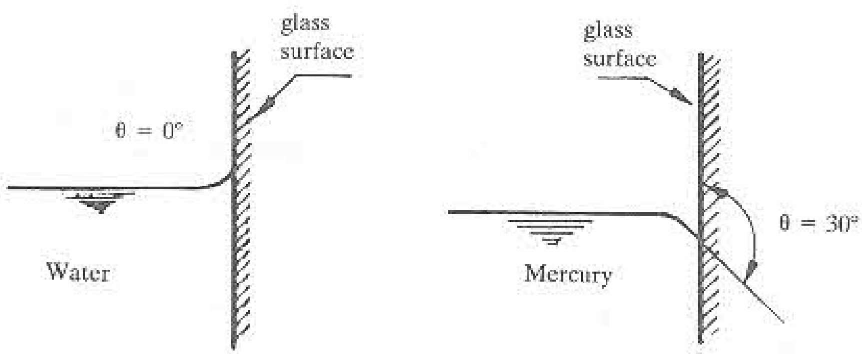



(a) Vertical capillary tube partially submerged in a liquid




(b) weted surface

(c) non-weted surface

*Fig. 1.3 Effect of surface tension on (a) capillary attraction (b) weted surface, and (c) non-weted surface*

## PROBLEMS ON CHAPTER ONE

**Problems on Sections 1 – 1 to 1 – 5**

1.1 Calculate the air pressure in an automobile tire of orginal pressure of 300 kPa abs at 25°C if the temperature becomes

a) – 10°C

b) 40°C

Assume no appreciable change in the volume of the tire.

1.2 A rigid tank of volume $0.05 m^3$ contains 0.5 kg air. What will be the pressure in the tank if the temperature becomes 30°C?

1.3 Air is maintained at atmospheric pressure (100 kPa abs). Calculate the density of air at

a) – 20°C

b) 0°C

c) 20°C

d) 40°C

1.4 Calculate the density of air at 0.2 MPa abs and temperatures of 30, 100, and 200°C.

1.5 Air is maintained in a balloon of volume $3 m^3$ at 150 kPa abs and 30°C. Calculate the volume of the balloon if the pressure is increased to 600 kPa abs while the temperature is kept constant.

**Problems on Sections 1 – 6 and 1 – 7**

1.6 Calculate the isothermal bulk modulus of air at atmospheric pressure and 30°C.

1.7 Calculate the required increase of pressure to reduce a given volume of water at atmospheric pressure by about 1%.

1.8 Calculate the bulk modulus of fluid A if its volume is reduced from 1 liter to 0.99 liters as the acting pressure is increased from $1 MN/m^2$ abs to $3 MN/m^2$ abs.

1.9 Five liters of water are maintained at atmospheric pressure and temperature of 20°C. If the pressure on the water is increased to 81 atmosphere, calculate the change in the volume of the water and its percentage of the original volume.

1.10 Calculate the speed of sound in the following mediums

a) Air at 30°C and $10^5$ Pa abs.

b) Helium at 30°C and $10^5$ Pa abs.

c) Air at 30°C and $10^5$ Pa abs.

d) Air at 300°C and $10^5$ Pa abs.

e) Water at 30°C and $10^5$ Pa abs.

**Problems on Section 1 – 8**

1.11 Water at 30°C flows over a solid wall with a velocity profile given by $u = 40\,y - 1.2 \times 10^4\,y^2$, m/s, where y is the distance from the wall in water. Calculate the shear stress on the wall.

1.12 Calculate the power required to rotate a 20 cm diameter shaft with constant angular speed of 2000 r/min inside a bearing of 20.06 cm in diameter and 50 cm long. The space between the shaft and the bearing contains oil of viscosity $\mu = 0.02$ kg/m.s.

1.13 Find the kinematic viscosity and dynamic viscosity of the following fluids

a) Water at 20°C and $10^5$ Pa abs.

b) Air at 20°C and $10^5$ Pa abs.

c) Helium at 20°C and $10^5$ Pa abs.

d) Kerosene at 20°C and $10^5$ Pa abs.

e) Mercury at 20°C and $10^5$ Pa abs.

1.14 The velocity profile of water flowing near a solid surface is given by the following equation

$u = 4\,y^{2/3}$ where u is the velocity in m/s and y is normal distance from the wall in m. Calculate the velocity gradient and the shear stress at y = 0, 0.2, 0.5 m

1.15 Calculate the approximate viscosity of the oil in Fig. (1.4)

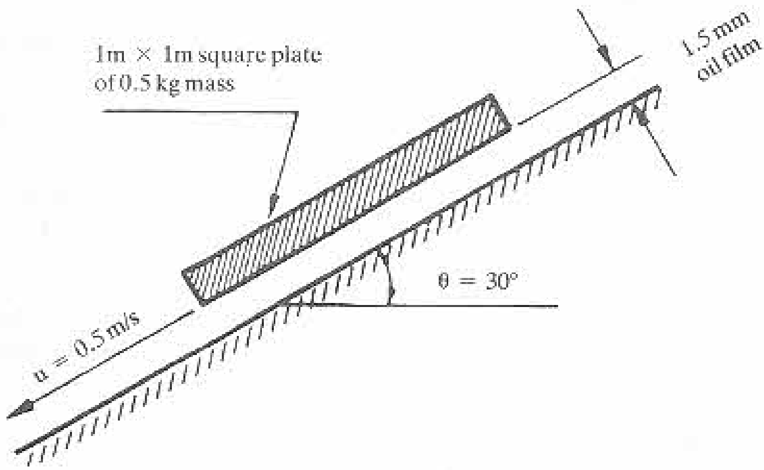



*Fig. 1.4*

**Problems on Section 1 – 9**

1.16 Calculate the pressure in a water droplet of diameter 0.05 mm at 30°C if the outside pressure is atmospheric. At same temperature and external pressure, what will be the internal pressure of the droplet if its diameter becomes 0.1 mm?

1.17 What is the diameter of a water droplet at 30°C when its internal pressure is about 5% greater than the outside pressure?

1.18 Calculate the maximum capillary rise of water (20°C) to be expected in a vertical glass tube 1mm in diameter.

1.19 Calculate the maximum capillary depression of mercury to be expected in a vertical glass tube 1mm in diameter at 15.5°C.

**General Problems on Chapter 1**

1.20 The velocity distribution of a pipe flow is given by

$$\frac{u}{U} = \left(1 - \frac{r}{R}\right)^2$$

where U is the maximum centerline velocity, R is the pipe radius and r is the distance from the centerline. If U = 10 m/s and R = 5 cm, calculate the shear stress on the wall if the flowing fluid is

a) Water at 20°C

b) air at 20°C and 100 kPa abs.

1.21 Oil with density of 850 $kg/m^2$ flows through a 10 cm diameter pipe. The shear stress at the pipe wall is measured as 3.2 Pa and the velocity profile is given by

$$u = 2 - 800\, r^2$$

where r is the radial distance from the pipe centerline in meters. Calculate the kinematic viscosity of the oil.

1.22 Derive the dimensions of the following quantities

a) vapour pressure

b) density

c) specific gravity

d) specific volume

e) dynamic viscosity

f) kinematic viscosity

g) surface tension

1.23 Air at 15°C and 0.15 MPa abs is compressed isentropically to half the original volume. Calculate the final pressure and temperature and the speed of sound in air before and after compression.

1.24 Derive the equation for the vertical capillary rise between vertical parallel plates.

1.25 A fluid is placed in the 0.5 mm clearance between two parallel plates. The lower plate is stationary while the upper one moves with uniform velocity of 0.5 m/s. Assuming linear variation of the fluid velocity, calculate the shear stress at the upper and lower plates if the fluid is

a) Water

b) Kerosene.

CHAPTER TWO
# FLUID STATICS

## 2-1- Introduction

This chapter deals with fluid statics, i.e. fluid at rest. Generally, the fluid can not resist any shearing force where the latter would result in fluid flow. Therefore, the forces on a fluid at rest must be perpendicular to any plane. This causes the pressure at any point in a static fluid continuum to be the same in all directions. In this chapter we shall derive the hydrostatic equation for the pressure gradient in the fluid under the effect of gravitational field. This equation is considered a good base to study many subjects such as measurement of pressure by manometers, forces due to hydrostatic pressure, buoyancy and static stability of floating bodies. These subjects are useful in studying many engineering problems such as forces on dams or any submerged body, pressure variation in the atmosphere and ships design.

## 2-2- Concept of Pressure

The pressure at a point in a fluid continuum is defined as the magnitude of the normal compression force per unit surface area surrounding the point. As stated before, a fluid in static equilibrium can not sustain shear stresses. Bearing this in mind, let us study the forces on an infinitesimal fluid element in static equilibrium as shown in Fig. (2.1). The figure is an arbitrary shape of a rectangular prism with base dimensions as given and a unit depth perpendicular to the base. At static equilibrium, the forces in the x and y directions are given as follows.

$$\Sigma F_x = P_y . dy - P_s . ds \, . \sin \alpha \tag{2.1a}$$

$$\Sigma F_y = P_x . dx - p_s . ds \, . \cos \alpha - \frac{\rho g \, dx \, dy}{2} \tag{2.1b}$$

where

$P_x$, $P_y$ and $P_s$ are the external pressures on the faces dx, dy and ds respectively,
$\rho$ is the density of the fluid and
w is the weight of the element $= \frac{\rho g . dx . dy}{2}$

From the geometry, we have $ds = \frac{dy}{\sin\alpha} = \frac{dx}{\cos\alpha}$

Substituting ds in Eqs. (2.1a) and (2.1b) by $\frac{dy}{\sin\alpha}$ and $\frac{dx}{\cos\alpha}$ respectively, then letting the size of the element tend to zero.

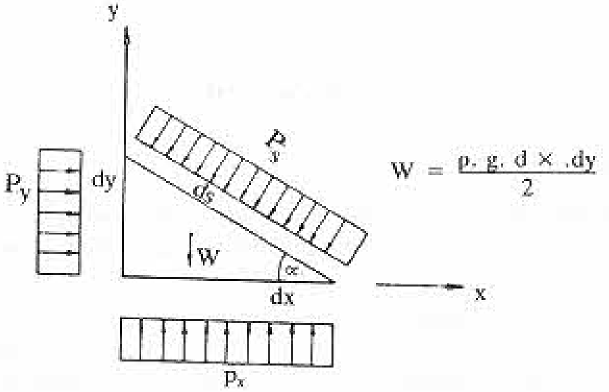



*Fig. 2.1 Equilibrium of forces on an infinitesmal fluid element*

These equations yield

$$P_x = P_y = P_s \tag{2.2}$$

This means that at any point in a static fluid continuum (i.e. no shear stresses) the pressure is the same in all directions.

## 2 – 3 – Pressure Variation in a Static Fluid

Let us consider the balance of forces on an infinitesimal fluid element in a graviational field. Assume that the element has an area dA and a height dz with dA normal to the z direction as shown in Fig. (2.2a). The distance z is measured from an arbitrary datum plane in a direction opposite to gravity. The external forces acting on the fluid element are illustrated in Fig. (2.2b) where P is the pressure on the bottom surface, $(P + \frac{\partial P}{\partial z} dz)$ is the pressure on the top surface and W is the weight of the element.

The net force in the z direction is

$$PdA - (P + \frac{\partial P}{\partial z} dz)\ dA - \rho g\ dA\ dz = 0 \tag{2.3}$$

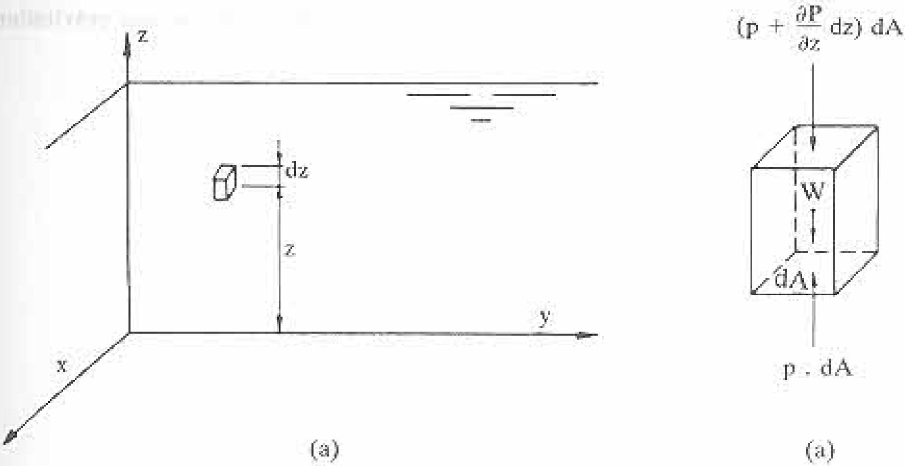



*Fig. 2.2 Balance of forces on an infinitesimal fluid element in a gravitational field*

Simplifying the previous equation yields

$$\frac{\partial P}{\partial z} = -\rho g \qquad (2.4a)$$

Similarily, the balances of forces in the x and y directions give

$$\frac{\partial P}{\partial x} = 0 \qquad (2.4b)$$

$$\frac{\partial P}{\partial y} = 0 \qquad (2.4c)$$

From Eq. (2.4) we conclude that the pressure is constant over a horizontal surface in a static fluid and it varies in the z direction only; therefore the total derivative replaces the partial derivative in Eq. (2.4a), thus

$$\frac{dP}{dz} = -\rho g \qquad (2.5)$$

This is known as the hydrostatic equation for the pressure gradient at a point in a fluid under the effect of gravitational field. This equation is a basic equation in fluid statics.

Knowing the values of ρ and g, the pressure distribution can be determined by integrating Eq. (2.5). This can be done for special distributions of ρ and g as follows.

2 – 3 – 1 – **Pressure variation in an incompressible fluid in a constant gravitational field**

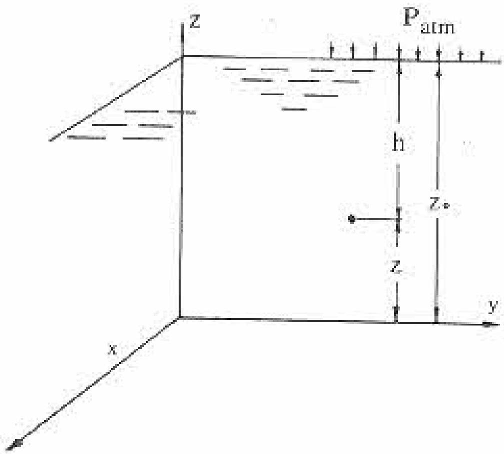



*Fig. 2.3 Incompressible fluid in constant gravitational field*

In this case both ρ and g are specified constants. Now, integrating Eq. (2.5) from any position z where the pressure is P to the fluid surface at position z where the pressure is that of atmosphere $P_a$ (see Fig. 2.3), so that

$$\int_{P}^{P_a} dP = -\int_{z}^{z_o} \rho g \, dz \tag{2.6}$$

i.e.

$$P - P_a = \rho g \, (z_o - z) \tag{2.7}$$

or

$$P_g = \rho g h \tag{2.8}$$

where

$h = (z_o - z)$ is the elevation head, and

$P_g = P - P_a$ is the gage pressure.

By definition,the gage pressure is the difference between the absolute pressure P and the atmospheric pressure $P_a$. It is important to know the physical meaning of the gage pressure because it is used quite often in engineering work. A better picture of this meaning can be gained from Fig. (2.4). It must be noted that

1. atmospheric pressure varies with weather and altitude
2. all absolute pressures are positive while gage pressures may be positive or negative.

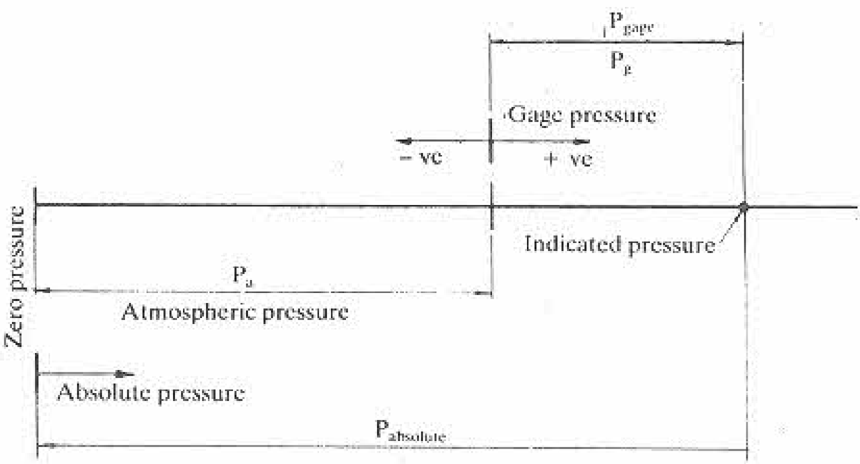



*Fig. 2.4 Absolute and gage pressures*

Also, Eq. (2.5) can be integrated to give

$$P + \rho g z = P_o \tag{2.9}$$

where $P_o$ is the pressure at $z = 0$. The value of $P_o$ is constant throughout a static fluid and it is equal to the sum of static pressure P and the pressure due to potential head $\rho g z$ at any location. Equation (2.9) can be rearranged to the following form

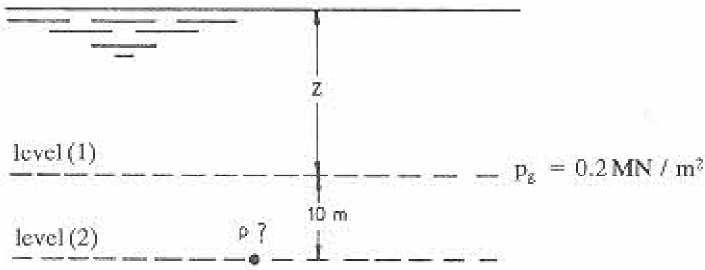
$$\frac{P}{\rho g} + z = \frac{P_o}{\rho g} \tag{2.10}$$

The terms $\frac{P}{\rho g}$ and z are the static pressure head and the potential head respectively.

**EXAMPLE 2 – 1**

The gage pressure at a horizontal level in a river is 0.2 $MN/m^2$. Determine the gage and absolute pressures at a point 10m below this level. Consider that atmospheric pressure and water density are 0.10 $MN/m^2$ and 1030 $kg/m^3$ respectively.

Problem Description




Data of the Problem

* atmospheric pressure = 0.1 MN/$m^2$
* Water density = 1030 kg/$m^3$
* $P_g$ at level (1) = 0.2 MN/$m^2$

Requirement

* The gage and absolute pressures at a point 10m below level (1)

Solution

From Eq. (2.5), we have

$P_{g1} = \rho g h_1$ at level (1), (I)

$P_{g2} = \rho g h_2$ at level (2), (II)

Subtracting I from II, gives

$P_{g2} = P_{g1} + \rho g (h_2 - h_1)$ (III)

Therefore

$P_{g2} = 0.2 \times 10^6 + 1030 \times 9.8 \times 10$

$= 0.30094 \times 10^6$ N/$m^2$

$= 0.30094$ MN/$m^2$ (IV)

To get the absolute pressure, $P_2$, we know that

$P_2 = P_{g2} + P_{atm}$, (V)

hence

$P_2 = 0.30094$ (MN/$m^2$) + 0.10 (MN/$m^2$)

$= 0.40094$ MN/$m^2$ (VI)

### 2-3-2- Pressure variation in a compressible fluid in a variable gravitational field

Let us consider a compressible fluid like a perfect gas obeying the law $P = \rho RT$. The hydrostatic equation, Eq. (2.5), applied to this field becomes

$$dP = -\frac{\rho g}{RT}\,dz \tag{2.11}$$

i.e.

$$\frac{dP}{P} = -\frac{g}{RT}\,dz \tag{2.12}$$

where:

g is the gravitational acceleration,
R is the gas constant and
T is the temperature of the fluid.

To integrate Eq. (2.12), the relation between g, T and z needs to be specified. In the atmosphere of the earth, the acceleration at any altitude is given by

$$g = g_o \frac{r^2}{(r + z)^2} \tag{2.13}$$

where:

r is the radius of the earth, $r \cong 6334$km, and
$g_o$ is the gravitational acceleration at the surface of the earth, $g_o \cong 9.81$ m/s$^2$.

Also the temperature (T) can be approximated as follows

$$T = T_o - Kz \tag{2.14}$$

where:

$T_o$ is the temperature at sea level and
K is constant known as the rate of temperature-lapse.

The rate of temperature-lapse is approximately equal to $6.6 \times 10^{-3}$ °C/m in the first layer of the atmosphere. This layer is usually called troposphere and its height is about 11km. Above the troposphere, there is a second layer that is more or less constant in temperature. This isothermal layer is called the stratosphere and has a temperature around −56°C. Its height is in the neighborhood of 9km beyond the troposphere. By substituting from Eqs. (2.13) and (2.14) into Eq. (2.12) we get

$$\frac{dP}{P} = -\frac{g_o r^2}{R} \cdot \frac{dz}{(r+z)^2 \cdot (T_o - kz)} \tag{2.15}$$

Integrating the above equation from the value $P_o$ at the point $z = 0$, sea level, to the value P at the point z, above sea level, we find

$$\int_{P_o}^{P} \frac{dP}{P} = -\frac{g_o r^2}{R} \cdot \int_0^z \frac{dz}{(r+z)^2 \cdot (T_o - kz)}$$

$$= -\frac{g_o r^2}{R} \cdot \left( \int_0^z \frac{k^2}{(kr+T_o)^2 \cdot (T_o - kz)} + \int_0^z \frac{kdz}{(r+z)(kr+T_o)^2} + \int_0^z \frac{dz}{(kr+T_o)(r+z)^2} \right) \tag{2.16a}$$

i.e.

$$\ln \frac{P}{P_o} = -\frac{kg_o}{R} \cdot \left(\frac{r}{kr+T_o}\right)^2 \cdot \left( \ln \frac{T_o(r+z)}{r(T_o - kz)} + \frac{z(kr+T_o)}{kr\ (r+z)} \right) \tag{2.16b}$$

However, if the temperature T and the gravitational acceleration g are taken to be constant-which is a reasonable approximation for most engineering works near the surface of the earth-Eq. (2.12) can be integrated between the same previous limits to give

$$\ln \frac{P}{P_o} = -\frac{g}{RT_o} z \tag{2.17}$$

Using the gas law $P = \rho RT$ to replace $RT_o$ by $\frac{P_o}{\rho_o}$ Eq. (2.17) becomes

$$\ln \frac{P}{P_o} = -\frac{g\rho_o}{P_o} z \tag{2.18}$$

therefore

$$P = P_o \exp \{-g(\rho_o/P_o)\ z\} \tag{2.19}$$

At sea level with $T = 20\ °C$, the values of g, $P_o$ and $\rho_o$ are 9.8 $m/s^2$, $1.01 \times 10^5$ and 1.2kg/$m^3$ respectively. Therefore Eq. (2.19) becomes

$$P = P_o \exp(-0.116 \times 10^{-3} z)$$

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**EXAMPLE 2 – 2**

If the rate of temperature-lapse in the troposphere is 6.6 °C per kilometre and the pressure is a function of the density according to the relation $P\rho^{-n}$ = constant, i.e. polytropic process. Determine the index n and express the pressure, density and temperature as functions of altitude and data at sea level.

Data of the Problem

* temperature-lapse rate (K) = 6.6 °C/km
* relation between the pressure and the density is considered $P\rho^{-n}$ = constant

Requirements

* The index n
* the pressure, density and temperature as functions of altitude

Solution

As given in the data of the problem the relation between pressure and density takes the form

$$P\rho^{-n} = \text{constant} \tag{I}$$

where n is the polytropic expansion exponent. Then the density can be given by

$$\rho = b \, . \, P^{\frac{1}{n}} \tag{II}$$

where b is a constant.
Substituting from Eq. (II) into Eq. (2.5), hence

$$\frac{dP}{dz} = -(b \, . \, P^{\frac{1}{n}}) \, g = - bg \, P^{\frac{1}{n}} \tag{III}$$

Integrate Eq. (III), from sea level denoted by subscript 0, to any other level, then rearrange using the equation of gas state to get

$$\frac{P}{P_o} = \{ 1 - \frac{g(n-1)}{nRT_o} z \}^{\frac{n}{n-1}} \tag{IV}$$

where R is the gas constant of air (R = 287 J/kg. K). From the polytropic relations and Eq. (IV), the density and temperature variations with the altitude are given by

$$\frac{\rho}{\rho_o} = \{ 1 - \frac{g\,(n-1)}{nRT_o} z \}^{\frac{1}{n-1}} \tag{V}$$

$$\begin{aligned} T &= T_o - \frac{g\,(n-1)}{n\,R} z \\ &= T_o - k\,z \end{aligned} \tag{VI}$$

where

$$k = \frac{g\ (n-1)}{n\ R} = \text{rate of temperature-lapse} \tag{VII}$$

Substituting from the data of the problem into Eq. (VII) instead of the temperature-lapse rate gives

$$n = 1.24 \tag{VIII}$$

Substituting from Eq. (VIII) into Eqs. (IV), (V) and (VI) by considering sea level conditions ($T_o$ = 293K), yields

$$T = 293 - 6.6\ z$$

$$\frac{P}{P_o} = (1 - 2.253 \times 10^{-5}\ z)^{5.167}$$

$$\frac{\rho}{\rho_o} = (1 - 2.253 \times 10^{-5}\ z)^{4.167}$$

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## 2 – 4 – Measurement of Pressure

Most of pressure measuring techniques are based on the simple concept of balancing the unknown pressure against a pressure due to either the gavitational field or a mechanical system. In the first case where the unknown pressure is balanced by the weight of a column of liquid in static equilibrium, the device used is called manometer. In the second case, a mechanical system is used to balance the pressure against a mechanical pressure. Such a device is widely known as pressure gage. Both manometers and one type of the pressure gages, Bourdon gage, are discussed below.

### 2 – 4 – 1 – Manometers

The simplest type of manometers is the open tube manometer shown in Fig. (2.5). It consists of a U-shaped tube containing a liquid which may be mercury, water, alcohol or any other liquid. One side of the tube is open to the atmosphere, while the other side is connected to the fluid whose pressure is being measured. The pressure at the bottom of the U-tube is of course the same, but on the left hand side of the tube it is given by

$$P + \rho_f\, g\, h_f + \rho_m\, g\, h_1 \tag{2.20}$$

While on the right hand side of the tube is given by

$$P_a + \rho_m g (h_m + h_l) \qquad (2.21)$$

where $\rho_f$ and $\rho_m$ are the densities of the fluid whose pressure is being measured and the manometer liquid respectively, so that

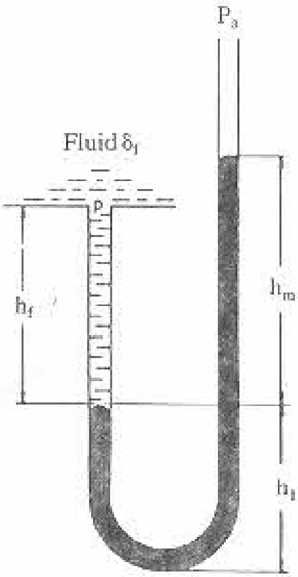


*Fig. 2.5 Open tube manometer*

$$P + \rho_f g h_f + \rho_m g h_l = P_a + \rho_m g(h_m + h_l) \qquad (2.22)$$

Therefore, the gage pressure is given as follows

$$P_g = P - P_a = g (\rho_m h_m - \rho_f h_f) \qquad (2.23)$$

For gases, densities are very small relative to manometer liquid density, hence may be stated that

$$P_g = \rho_m g h_m \qquad (2.24)$$

Another U-shaped tube configuration is shown in Fig. (2.6). Considering Fig. (2.6a), the pressure P at any point in the field is given by the following relation

$$P = P_a + g (\rho_f h_f + \rho_m h_m) \qquad (2.25)$$

The second term on the right hand side of the above equation is the gage pressure $P_g$, i.e.

$$P_g = g\,(\rho_f\,h_f + \rho_m\,h_m) \qquad (2.26)$$

Figure (2.6b) also gives the gage pressure $P_g$ as

$$P_g = g\,(\rho_f\,h_f - \rho_m\,h_m) \qquad (2.27)$$

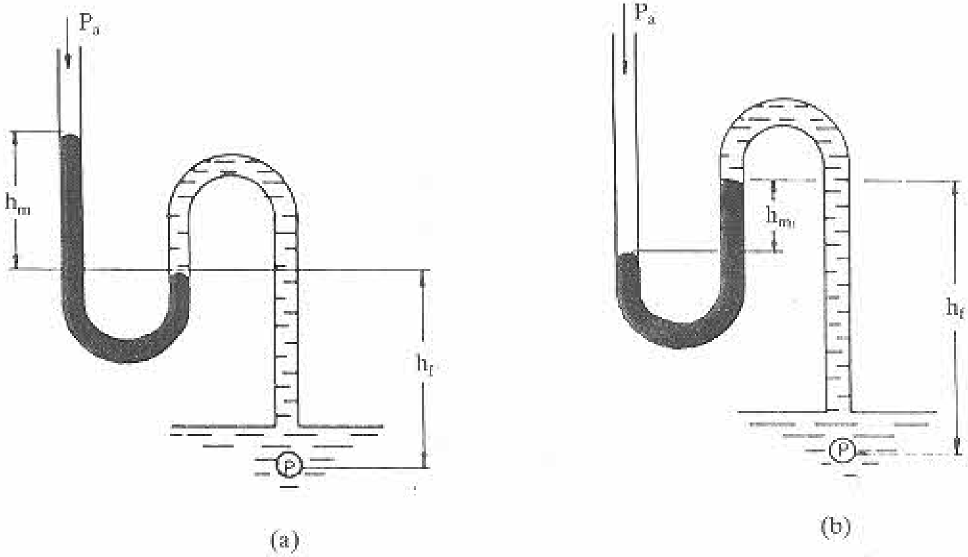


*Fig. 2.6 Another configuration of open tube manometer*

In some cases, the fluid of the conduit for which the pressure is to be measured is also used as a manometer liquid and that usually occurs at moderate pressures. A good arrangement for this case is to use a vertical tube tapped to the conduit surface. This arrangement is called "piezometer", (Fig. 2.7). In this case the gage pressure is

$$P_g = \rho_f\,g\,h_f \qquad (2.28)$$

Another arrangement for the U-shaped tube manometer is the well-type shown in Fig. (2.8). The tube of the manometer is connected to an open reservoir having cross sectional area $A_2$; that is substantialy greater than the area of the tube $A_1$. The zero reading of the manometer is usually taken at the level of the manometer fluid with atmospheric pressure on both sides of the manometer fluid. When applying pressure to the manometer, new settings of the manometer liquid surfaces will be establised in both the tube and in the reservoir. In Fig. (2.8) the distance between the surface of the manometer liquid in the tube and that in the reservoir is denoted by $h_m$ and that between the surface of the manometer liquid in the tube and the zero reading is denoted by h. The relation between h and $h_m$ is given by

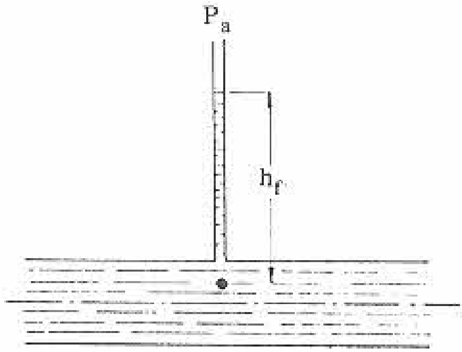


*Fig. 2.7 Piezometer*

$$h A_1 = (h_m - h) A_2 \qquad (2.29)$$

Substituting by the value of $h_m$ from Eq. (2.29) into Eq. (2.24) we get

$$P_g = \rho_m g h \left(1 + \frac{A_1}{A_2}\right) \qquad (2.30)$$

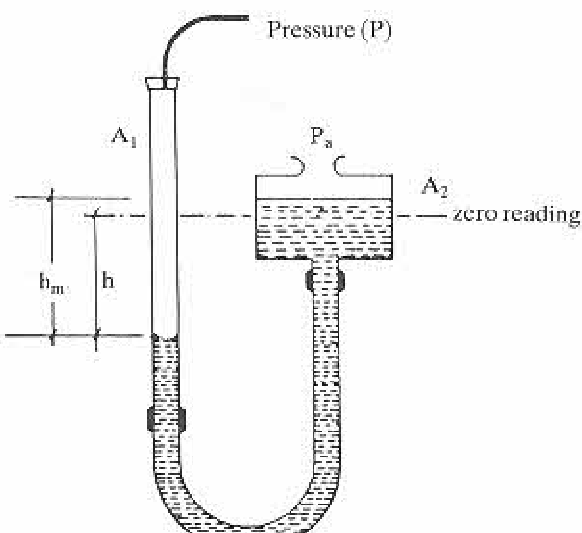


*Fig. 2.8 Well-type manometer*

Comparison of Eqs. (2.30) and (2.24) suggests that referring the manometer reading to the zero reading (i.e. using h instead of $h_m$ in Eq. 2.24) would only be an approximation. The accuracy of this approximation depends on the value of $A_1/A_2$. The term $(1 + A_1/A_2)$ is known as the correction factor and is usually taken into consideration when marking the scale of the manometer so that the correct reading can be taken directly from the manometer scale.

When the U-shaped manometer is used to measure the pressure difference between two different level locations, see Fig. (2.9), the manometer is known as differential manometer. To calculate the pressure difference, one follows the tube, starting from one location to the other, keeping in mind that the pressures are equal over horizontal planes within continuous columns of the same fluid.

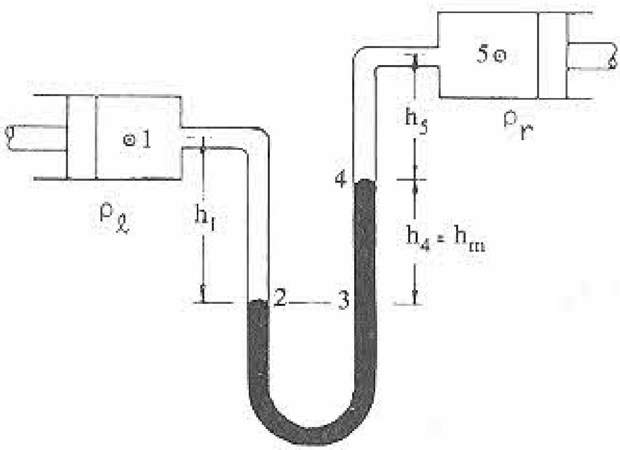


*Fig. 2.9 Differential manometer*

If $\rho_\ell$, $\rho_m$ and $\rho_r$ are the densities of fluids in the left hand side location, the manometer and the right hand side location respectively, the values of pressure at points 2 and 3-being equal-can be expressed as follows

$$P_3 = P_2 = P_1 + \rho_\ell g h_1 \qquad (2.31)$$

While the pressure at point 4 can be expressed as

$$P_4 = P_3 - \rho_m g h_4 \qquad (2.32)$$

Substituting $P_3$ from Eq. (2.31) gives

$$P_4 = P_1 + \rho_\ell g h_1 - \rho_m g h_4 \tag{2.33}$$

Similarly, the pressure at point 5 is

$$P_5 = P_4 - \rho_r g h_5 \tag{2.34}$$

Replacing the value of $P_4$ from Eq. (2.33) yields

$$P_1 - P_5 = g(\rho_m h_4 + \rho_r h_5 - \rho_\ell h_1) \tag{2.35}$$

This result can be derived directly using a simple rule by following the tube of the manometer from one side of the manometer assembly to the other through steps. Each step manipulates one fluid at a time where the pressure difference can be calculated easily. Applying this to Fig. (2.9) it would be found that

$$\begin{aligned} P_1 - P_5 &= (P_1 - P_2) + (P_2 - P_4) + (P_4 - P_5) \\ &= (-\rho_\ell g h_1) + (\rho_m g h_4) + (\rho_r g h_5) \\ &= g(\rho_m h_4 + \rho_r h_5 - \rho_\ell h_1) \end{aligned} \tag{2.36}$$

i.e., the same conclusion as Eq. (2.35). This rule is general and can be applied to any manometer assembly, see EXAMPLE 2 – 4. If $\rho_r$ and $\rho_\ell$ are negligible with respect to $\rho_m$, a good approximation of the above equation provides

$$P_1 - P_5 = \rho_m g h_4 = \rho_m g h_m \tag{2.37}$$

The differential manometer can be used to measure accurately small pressure differences by using two immiscible liquids of slightly different densities in the U-tube of the manometer.

### 2 – 4 – 2 – **Bourdon gage**

Bourdon gage, Fig. (2.10), is the most widely used type of industrial pressure gages. The pressure to be measured is applied to the inside of a phosphor bronze tube of flattened elliptical cross section. The tube is bent into an arc or coil. One end of the tube is held rigidly while the other is free to move inward or outward. The movement of the free end rotates a pointer through a linkage and gear arrangement. The applied pressure tends to change curvature of the coil. In doing this, the quadrant gear rotates the central pinion which in turn rotates the attached pointer. The movement of the free end depends upon the difference in the inside pressure and the outside pressure, the latter being the atmospheric pressure. Thus the pointer rotation, representing the gage pressure, is read on a scale. This scale normally reads zero when the gage is open to atmosphere. The scale is calibrated to read the gage pressure directly. Many of bourdon gages are calibrated to read pressure both above and below atmospheric pressure, the latter is usually known as vacuum.

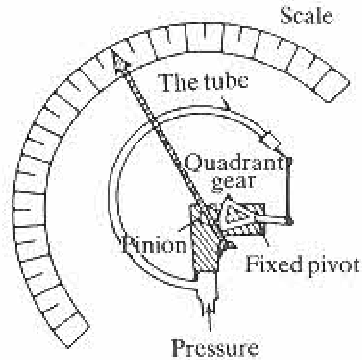


*Fig. 2.10 Bourdon gauge*

**EXAMPLE 2 – 3**

In Fig. (2.6a), the manometer liquid and the fluid are mercury and water respectively. If $h_m$ = 30cm and the height h from the top of the mercury to the centre of the fluid pipe is 45cm. Calculate the gage pressure at the pipe centreline in Newton per meter square, centimeters of mercury and meters of water

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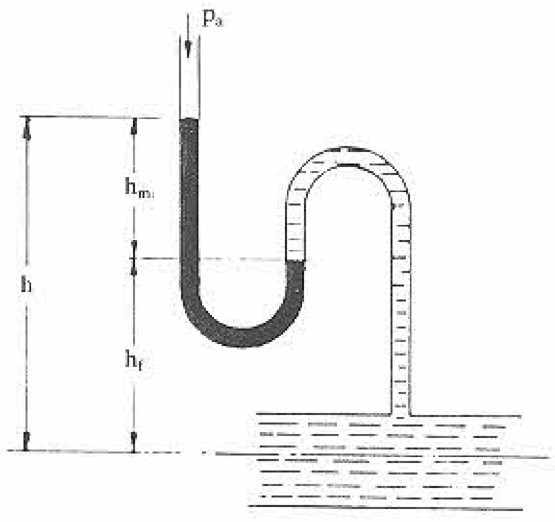
Problem Description



Data of the Problem

* the manometer liquid is mercury
* the fluid is water
* $h_m$ = 30 cm
* h = 45cm (the height from the top of the mercury to pipe centre).

Requirement

* the gage pressure at the pipe centreline in $N/m^2$, cms of mercury (Hg) and cms of water (w).

Solution

Using Eq. (2.27), the gage pressure becomes

$$P_g = (\rho_f h_f + \rho_m h_m), \qquad (I)$$

$$h_f = h - h_m = 0.45 - 0.30 = 0.15 \qquad (II)$$

Substituting from Eq. (II) into Eq. (I), yields

$$P_g = 9.8\,(1000 \times 0.15 + 13600 \times 0.3)$$

$$= 41.454 \times 10^3 \quad N/m^2 \qquad (III)$$

The gage pressure $P_g$ is also equal to $\rho_f\, g\, h_f$ where f refers to any fluid, thus

$$h_{Hg} = \frac{41.454 \times 10^3}{13.6 \times 10^3 \times 9.8} \times 100$$

$$= 31.103 \text{ cm mercury}$$

$$h_w = \frac{41.454 \times 10^3}{10^3 \times 9.8}$$

$$= 4.23 \text{ meters of water}$$

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**EXAMPLE 2 – 4**

An inclined manometer is connected to a reservoir containing water and gas having the dimensions shown in problem description. Calculate the pressure of the gas above the water in the following units:

a) $MN/m^2$, b) bars, c) atms, d) cm of mercury, e) meters of water. Take atmospheric pressure equal to 0.1013 $MN/m^2$.

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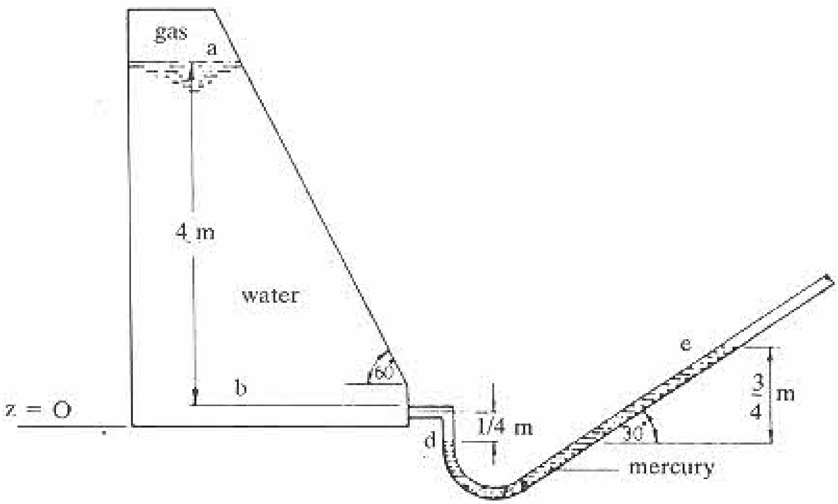
Problem Description




Data of the Problem

* see problem description

Requirements

* calculate the pressure of the gas in the following units:

a) $MN/m^2$, b) bars, c) atms, d) cm of mercury, e) meters of water

Solution

a) Integrating Eq. (2.5) between any two levels 1 and 2 gives

$$P_1 - P_2 = \rho g (z_2 - z_1) \quad \text{(I)}$$

The pressure difference between points "a" and "e" is given as follows

$$P_1 - P_4 = (P_1 - P_2) + (P_2 - P_3) + (P_3 - P_4)$$

Using Eq. (I), the pressure difference can be estimated as follows

$$P_1 - P_4 = -(1000 \times 9.8 \times 4) - (1000 \times 9.8 \times \frac{1}{4}) + $$
$$+ (13600 \times 9.8 \times \frac{3}{4})$$
$$= 0.05831 \times 10^6 \quad N/m^2 \qquad (II)$$

But $P_4 = 0.1013 \times 10^6$ N/m$^2$, therefore

$$P_1 = 0.05831 \times 10^6 + 0.1013 \times 10^6 = 0.15961 \times 10^6 \ N/m^2$$
$$= 0.15961 \ MN/m^2 \qquad (III)$$

b) Value of $P_1$ in bars, where bar = 0.1 MN/m$^2$

$$P_1 = \frac{0.15961}{0.1} = 1.5961 \quad bar$$

c) Value of $P_4$ in atms, where atm = 0.101325 MN/m$^2$

$$P_1 = \frac{0.15961}{0.101325} = 1.57523 \quad atm$$

d) Value of $P_1$ in cm of mercury
Refering to EXAMPLE (2 – 3),

$$P_1 = \frac{0.15961 \times 10^6}{13.6 \times 10^3 \times 9.8} \times 100 = 119.755 \ \text{cm mercury}$$

e) Value of $P_1$ in meters of water
Refering to EXAMPLE (2.3),

$$P_1 = \frac{0.15961 \times 10^6}{10^3 \times 9.8} = 16.287 \quad \text{meter of water}$$

---

## 2 – 5 – Forces due to Hydrostatic Pressure

In many engineering problems, the need arises that the resultant force due to hydrostatic pressure on a surface be calculated, and as well its line or point of action be identified. The surface under pressure may be curved or flat. Also, the pressure in the fluid field may be constant or variable. In the following analysis the general case will be considered first, then examples of its application under specified conditions will follow.

The total force "F" due to hydrostatic pressure, acting on a general surface submerged in a static fluid as shown in Fig. (2.11), may be estimated by evaluating its components $F_x$, $F_y$ and $F_z$. The following analysis is carried out for one component, but the other two components can be found by analogy.

On an infinitesimal area dA whose normal unit vector n is at an angle $\theta$ with the x-axis, the x-component of pressure forces is

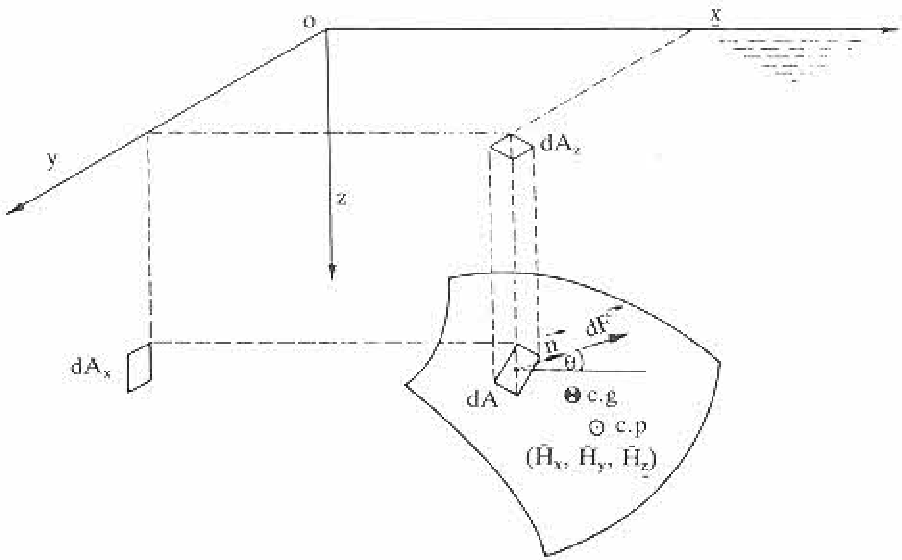



*Fig. 2.11 General surface submerged in a static fluid*

$$dF_x = dF \cos\theta = P\, dA \cos\theta \tag{2.38}$$

The term dA cos θ is the projection of the infinitesimal area onto y-z plane and is denoted hereafter by $dA_x$; so that

$$dF_x = P\, dA_x \tag{2.39}$$

Therefore

$$F_x = \int P\, dA_x \tag{2.40}$$

The moments of the x-component of the pressure forces about the y and z axes, $M_{y,x}$ and $M_{z,x}$ respectively, can be calculated as follows

$$dM_{y,x} = d\,F_x.z \tag{2.41}$$

$$dM_{z,x} = d\,F_x.y \tag{2.42}$$

Substituting by Eq. (2.40) into Eqs. (2.41) and (2.42), gives

$$dM_{y,x} = P.dA_x.z \tag{2.43}$$

$$dM_{z,x} = P.dA_x.y \tag{2.44}$$

Integrating Eqs. (2.43) and (2.44) yields the following

$$M_{y,x} = \int P.z.dA_x \tag{2.45}$$

$$M_{z,x} = \int P.y.dA_x \tag{2.46}$$

The line of action of the resultant pressure force in x-direction intersects the y-z plane at a point with coordinates ($\bar{H}_y$, $\bar{H}_z$), where

$$\bar{H}_y = \frac{M_{z,x}}{F_x} \tag{2.47}$$

$$\bar{H}_z = \frac{M_{y,x}}{F_x} \tag{2.48}$$

Substitution from Eqs. (2.40), (2.45) and (2.46) into Eqs. (2.47) and (2.48) yields

$$\bar{H}_y = \frac{\int P.y.dA_x}{\int P.dA_x} \tag{2.49}$$

$$\bar{H}_z = \frac{\int P.z.dA_x}{\int P.dA_x} \tag{2.50}$$

Note that the coordinates $\bar{H}_y$ and $\bar{H}_z$ represent the projection of the centre of pressure onto y-z plane.

The other two components of the resultant pressure force can be evaluated in the same manner along with the centre of pressure.

Attention needs to be drawn here to the fact that irrespective of the curvature of the surface, the net component of pressure forces in any direction can be found by projecting the surface on a plane perpendicular to this specified diréction, and calculating the pressure forces acting on the projected area. Also, the coordinates of the centre of pressure can be identified by finding the coordinates of the centre of pressure on the projected area.

Now we will consider the application of the above principles in the following cases.

### 2 – 5 – 1 – Application to a constant pressure field

If P is constant throughout the fluid field, Eq. (2.40) gives

$$F_x = P \int dA_x = P.A_x \tag{2.51}$$

where $A_x$ is the projection of the surface area onto y-z plane. Also, from Eqs. (2.45), (2.46), (2,49) and (2.50) we have

$$M_{y,x} = P \int_A z.dA_x \tag{2.52}$$

$$M_{z,x} = P \int_A y.dA_x \tag{2.53}$$

$$\bar{H}_y = \frac{\int_A y.dA_x}{A_x} \tag{2.54}$$

$$\bar{H}_z = \frac{\int_A z.dA_x}{A_x} \tag{2.55}$$

Analysing other components of forces in the same manner, it will be found that: "if the pressure acting on an area is constant all over this area then the centre of pressure coincides with the centre of area".

## 2 – 5 – 2 – Application to a variable pressure field

An example of this is when the pressure is a linear function of the vertical coordinate "z", i.e.

$$P = \rho g z \tag{2.56}$$

where $\rho$ is the density of the fluid and g is the gavitational acceleration, both considered constants. Substituting for P from Eq. (2.56) into Eq. (2.40) yields

$$\begin{aligned} F_x &= \int_A \rho g z \; dA_x \\ &= \rho g \int_A z \; dA_x \end{aligned} \tag{2.57}$$

but $\int z \; dA_x$ is the first moment of the projected area $A_x$ about y axis and can be defined as follows

$$\int z \, dA_x = A_x \bar{z} \tag{2.58}$$

where $\bar{z}$ is the z-coordinate of the centre of area. Therefore, the resultant force component in the x-direction can be evaluated by the formula

$$F_x = \rho g \, A_x \, \bar{z} \tag{2.59}$$

If the same procedure is followed with $M_{y,x}$ and $M_{z,x}$ it will be found that

$$M_{y,x} = \rho g \int_A z^2 . dA_x = \rho g \, I_{yy,x} \tag{2.60}$$

$$M_{z,x} = \rho g \int_A yz . dA_x = \rho g \, I_{yz,x} \tag{2.61}$$

Where $I_{yy,x}$ is the moment of inertia of the projected area $A_x$ with respect to the y axis and $I_{yz,x}$ is the product of inertia of the same area with respect to the y and z axes.

The coordinates of the line of action of the resultant pressure force in the x-direction are given by

$$\bar{H}_y = \frac{M_{z,x}}{F_x} = \frac{I_{yz,x}}{A_x . \bar{z}} \tag{2.62}$$

$$\bar{H}_z = \frac{M_{y,x}}{F_x} = \frac{I_{yy,x}}{A_x . \bar{z}} \tag{2.63}$$

But the values of $I_{yz,x}$ and $I_{yy,x}$ may be given as follows

$$I_{yz,x} = \bar{I}_{yz,x} + \bar{y} . \bar{z} . A_x \tag{2.64}$$

$$I_{yy,x} = \bar{I}_{yy,x} + z^{-2} . A_x \tag{2.65}$$

Where $\bar{I}_{yz,x}$ is the product of inertia with respect to the centroidal axes $\bar{y}$ and $\bar{z}$ ; $\bar{I}_{yy,x}$ is the moment of inertia with respect to the centroidal axis $\bar{y}$. Substituting from Eqs. (2.64) and (2.65) into Eqs. (2.62) and (2.63), we get

$$\bar{H}_y = \bar{y} + \frac{\bar{I}_{yz,x}}{A_x . \bar{z}} \tag{2.66}$$

$$\bar{H}_z = \bar{z} + \frac{\bar{I}_{yy,x}}{A_x . \bar{z}} \tag{2.67}$$

It is worth mentioning here that the resultant pressure force in the z-direction evaluated by the above procedure is given by

$$F_z = \rho g \int z \, dA_z \tag{2.68}$$

which is equal to the weight of the vertical cylindrical body of liquid with area A as base and the free surface of the liquid as top.

**Special case**

Consider the hydrostatic forces on a flat plate submerged in a static incompressible fluid. For simplicity the coordinates system can be selected in such a way that the submerged surface is perpendicular to the y-plane as shown in Fig. (2.12).

The hydrostatic force on the flat plate can be calculated from Eq. (2.61) as follows

$$F_x = \rho g . A_x . \bar{z} = \rho g \; A \sin\theta \; . \; \bar{z} \tag{2.69}$$

Therefore,

$$F = \frac{F_x}{\sin\theta} = \rho g \; A\bar{z} \tag{2.70}$$

The coordinates of the centre of pressure can be defined using Eqs. (2.66) and (2.67).

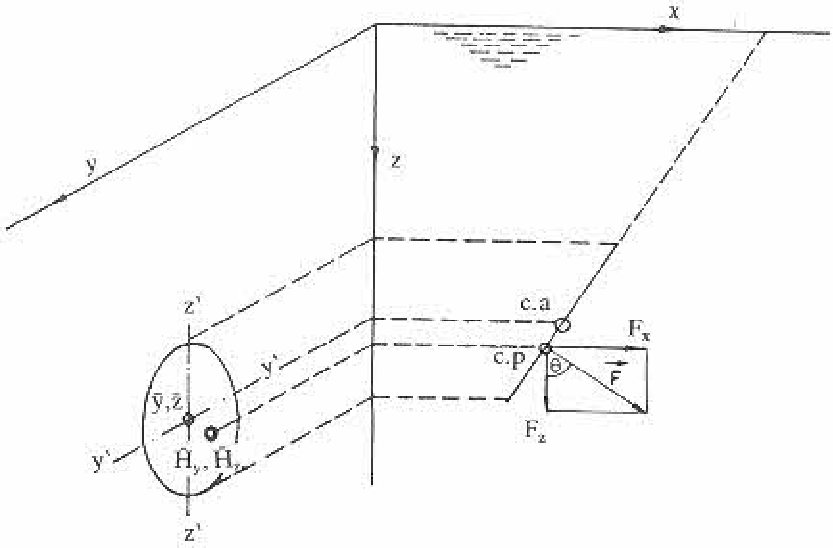



*Fig. 2.12 Flat plate submerged in a static incompressible fluid*

Another approach may be followed to solve this particular problem by assuming a $\xi$-$\zeta$ plane to be that of the flat plate as shown in Fig. (2.13) in which case the $\xi$-axis coincides with the free surface. In the new coordinate system, the net hydrostatic force on the plate and the coordinates of its centre of pressure can be determined using equations similar to Eqs. (2.69), (2.66) and (2.67) which give

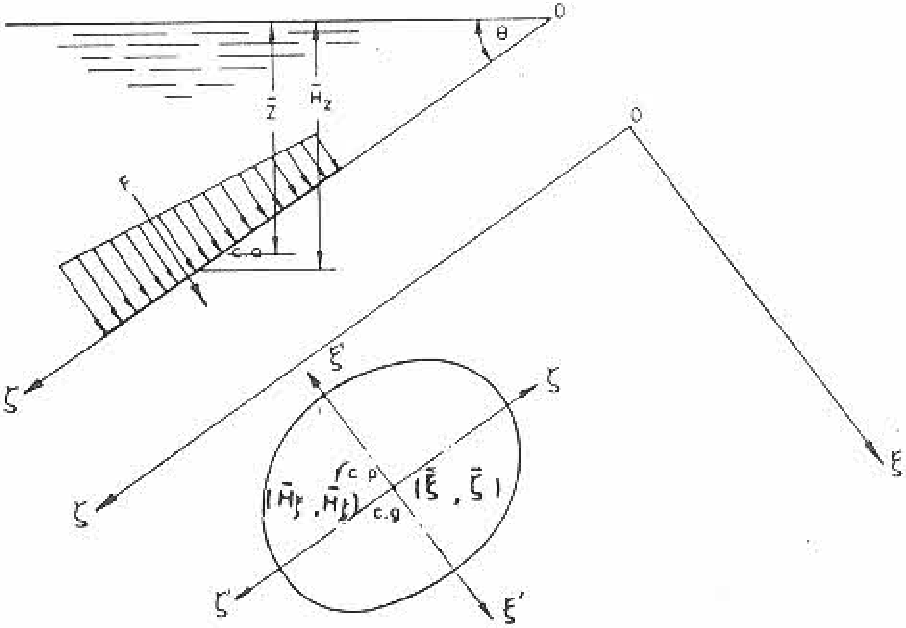



*Fig. 2.13 Plane submerged in a static incompressible fluid*

$$F = \rho g \cdot A\bar{z} \tag{2.71}$$

$$\bar{H}_\xi = \bar{\xi} + \frac{\bar{I}_{\xi\zeta}}{A} \tag{2.72}$$

$$\bar{H}_\zeta = \bar{\xi} + \frac{\bar{I}_{\xi\xi}}{A\bar{\zeta}} \tag{2.73}$$

where $\bar{\zeta}$ is defined from Fig. (2.13) as follows

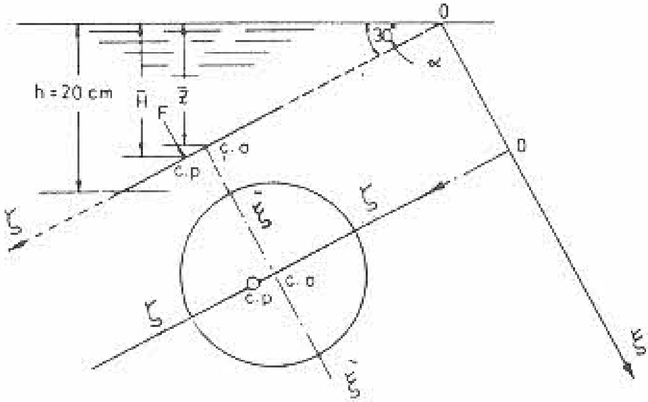
$$\bar{\zeta} = \frac{\bar{z}'}{\sin\theta} \tag{2.74}$$

**EXAMPLE 2 – 5**

A circular plate of diameter 20 cm is immersed in water as shown in the problem description. Calculate the magnitude and location of the hydrostatic force acting on one side of the plate.

---

Problem Description




Data of the Problem
 * circular plate of diameter d = 20 cm = 0.2 m
 * h = 20 cm = 0.2 m
 * $\alpha$ = 30°
 * density of the water $\rho$ = 1000 kg/m$^3$

Requirement
 * calculate F and $\overline{H}$

Solution

$$\bar{z} = 0.2 - 0.1 \sin 30^\circ$$
$$= 0.2 - 0.05$$
$$= 0.15 \text{ m} \qquad \text{(I)}$$

Taking the axes as shown in the problem description, and using Eqs. (2.71), (2.72) and (2.73), give

$$F = \rho g A \bar{z} = 1000 \times 9.8 \times (\pi \times 0.1) \times 0.15$$
$$= 46.18 \text{ N} \qquad \text{(II)}$$

$$\overline{H}_{\xi} = \bar{\xi} + \frac{I_{\bar{\xi}\bar{\zeta}}}{A\bar{\xi}} \; , \qquad \text{(III)}$$

but $\bar{\xi}$ = 0 and $I_{\bar{\xi}\bar{\zeta}}$ = 0 (due to symmetry), therefore

$$\bar{H}_{\xi} = 0 \qquad \text{(IV)}$$

$$\bar{H}_{\xi} = \bar{\xi} + \frac{I_{\xi\xi}}{A\bar{\xi}}$$

where

$$\bar{\xi} = \frac{\bar{z}}{\sin 30} = 0.15 \times 2 = 0.3 \text{ m} \qquad \text{(V)}$$

$$I_{\xi\xi} = \frac{\pi d^4}{64} = \frac{\pi (0.2)^4}{64} = 0.785 \times 10^{-4} \text{ m}^4 \qquad \text{(VI)}$$

thus

$$\bar{H}_{\xi} = 0.3 + \frac{0.785 \times 10^{-4}}{\pi (0.1)^2 \times 0.3} = 0.3083 \text{ m} \qquad \text{(VII)}$$

But $\bar{H} = \bar{H}_{\xi} \, . \, \sin 30$

$$\bar{H} = 0.3083 \times \frac{1}{2} = 0.15415 \text{ m} \qquad \text{(VIII)}$$

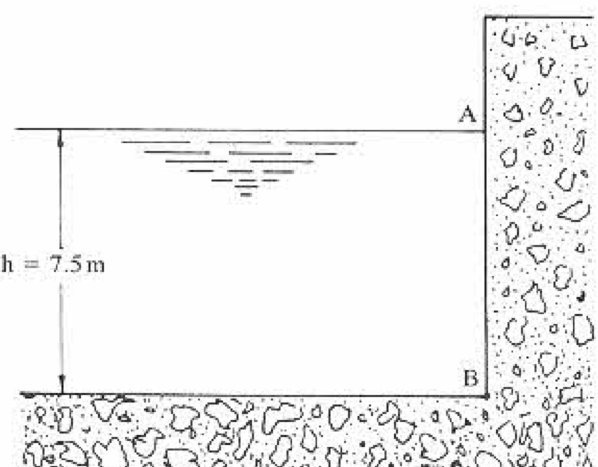
---

**EXAMPLE 2 – 6**

Calculate the resultant hydrostatic force of water, acting on the wall AB shown in the Problem Description with height 7.5m and width 5m.

---

Problem Description



Data of the Problem

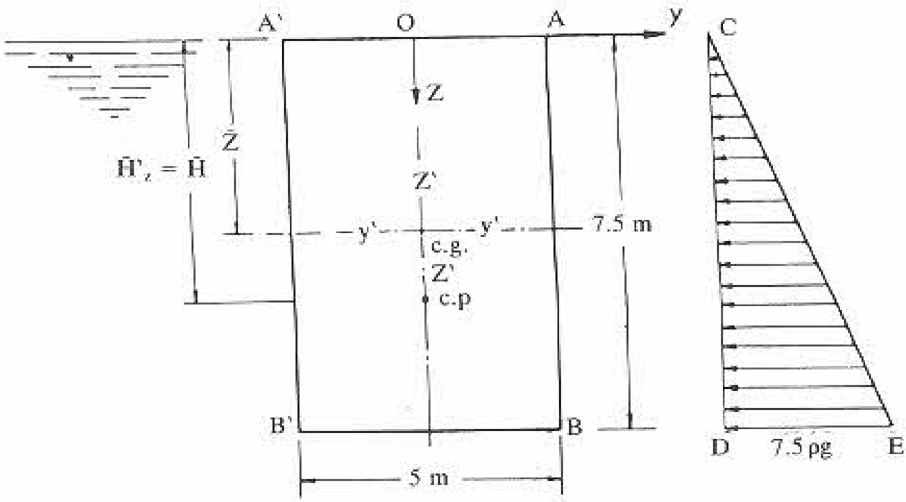
* AB is a vertical surface having:
  height, h = 7.5 m and
  width, W = 5.0 m
* density of water $\rho$ = 1000 kg/m$^3$

Requirement

* calculate the resultant hydrostatic force F and its location

Solution

Taking the axes at the centre of the plane AB as shown and using Eqs. (2.74) and (2.75), give




$$F = \rho g \; A\bar{z} = 1000 \times 9.8 \times (7.5 \times 5) \times \frac{7.5}{2} \qquad \text{(I)}$$

$$= 1378125 \quad N$$

From symmetry $\bar{H}_y$ = O, but $\bar{H}_z$ is given by

$$\bar{H}_z = \bar{z} + \frac{I_{\bar{y}\bar{y}}}{A\bar{z}} = \frac{7.5}{2} + \frac{5 \times (7.5)^3}{12 \times 5 \times 7.5 \times \frac{7.5}{2}} \qquad \text{(II)}$$

$$= \frac{7.5}{2} + \frac{7.5}{6} = \frac{2}{3}(7.5)$$

$$= 5 \quad m \qquad \text{(III)}$$

The wall is vertical, therefore

$$\bar{H} = \bar{H}_z = 5 \quad m$$

The same results can be obtained by considering the distribution of the hydrostatic pressure on the wall. At any point of this distribution the pressure is given by the relation $P = \rho gh$. Both $\rho$ and g being constants, the relation between P and h becomes a straight line, so that P is equal to zero at the free surface of the water and equal to $\rho$ gh ($1000 \times 9.8 \times 7.5$) at the bottom of the wall, with a linear distribution as shown in Fig. (I).

Thus the magnitude of the resultant force on the wall is equal to the area of the triangle CDE of Fig (I) multiplied by the width of the wall. Also the resultant force is in the same direction of the pressure distribution and acting at its centre of gravity.

So that

$$F = \frac{7.5\rho g \times 7.5}{2} \times 5 = 1378125 \text{ N} \qquad \text{(IV)}$$

which is the same result as Eq. (I)

It is obvious that the location of the resultant force, $\bar{H}$, is given as follows

$$\bar{H} = \frac{2}{3} \times 7.5 = 5 \quad m$$

and lies on the z axis due to symmetry, which is the same result as Eq. (III).

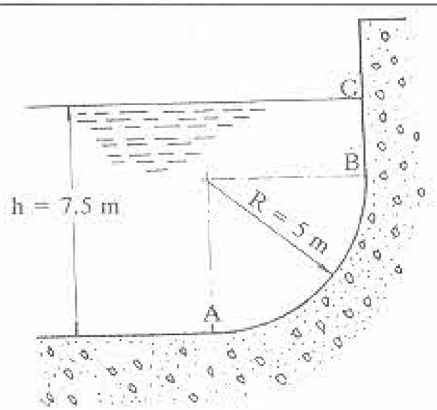
---

**EXAMPLE 2 – 7**

Calculate the resultant hydrostatic forces of water acting on 5m width of the wall ABC shown in the Problem Description.

---

Problem Description



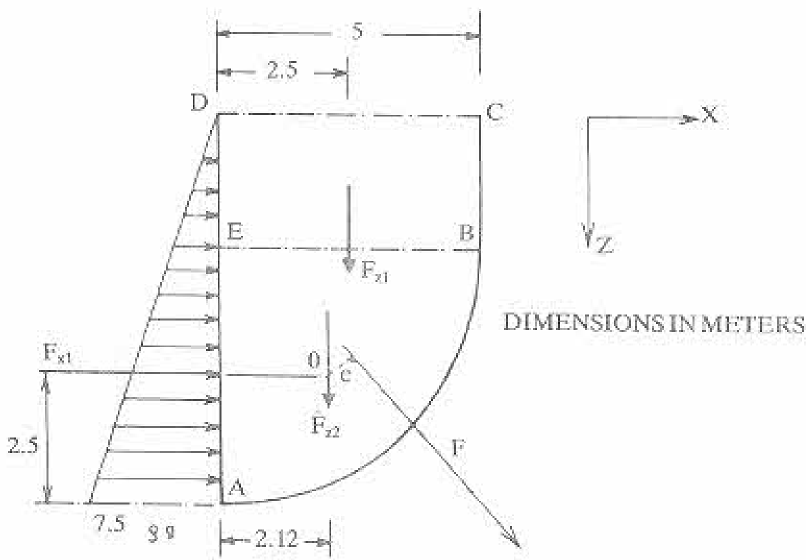
Data of the Problem

* AB is a quarter of a circular cylinder of R = 5m
* h = 7.5 m
* density of water = 1000 kg/m$^3$
* width of the surface w = 5m

Requirement

* calculate the resultant hdyrostatic force F and its location.

Solution




In the present problem the resultant force on the wall ABC has two components: in the x-and z-directions. The force per unit width in the z-direction is due to the weight of water in the volume having base area ABCD and one meter width. The force $F_z$ can be divided into two components: $F_{z1}$ which is the weight of the paral-lepiped BCDE with unit width and $F_{z2}$ which is the weight of the quarter cylinder ABE with unit width. The line of action of each of $F_{z1}$ and $F_{z2}$ passes through the centre of mass of their respective volumes (see Appendix A).
The value of $F_{z1}$ and $F_{z2}$ are as follows

$$F_{z1} = \text{(volume BCDE)} \times \text{(weight of unit volume of the fluid, i.e. } \rho g)$$
$$= (5 \times 2.5 \times 1) \times \rho g$$
$$= 12.5 \times \rho g \text{ N/m}$$
$$= 122500 \text{ N/m} \qquad \text{(I)}$$
$$F_{z2} = \text{(volume ABE)} \times (\rho g)$$

$$= \left(\frac{\pi \times 5^2}{4} \times 1\right) \times \rho g = 19.6 \times \rho g \quad \text{N/m}$$
$$= 192080 \quad \text{N/m} \qquad \text{(II)}$$

The horizontal component $F_x$ can be calculated directly from the hydrostatic pressure distribution as follows

$$F_x = \left(\frac{7.5 \times \rho g}{2} \times 7.5\right) \times 1 = 28.125 \times \rho g \text{ N/m}$$
$$= 275625 \quad \text{N/m} \qquad \text{(III)}$$

and its location at a distance 2.5 m of A
The resultant force per one meter width, F, is given as follows

$$F = \sqrt{F_x^2 + F_z^2}$$
$$= \sqrt{(28.125\,\rho g)^2 + (12.5\,\rho g + 19.6\,\rho g)^2}$$
$$= 42.68 \times \rho g \text{ N/m} \qquad \text{(IV)}$$
$$= 418264 \text{ N/m}$$

Total resultant force, Q, i.e. resultant force for 5 meters width

$$Q = F \times 5 = 418264 \times 5$$
$$= 2091320 \text{ N}$$
$$= 2.09 \text{ MN}$$

and its direction $\theta$ with respect to x-axis, is given by

$$\tan\theta = \frac{F_x}{F_y} = \frac{12.5\ \rho g + 19.6\ \rho g}{28.125} = 1.1413$$

i.e.
$$\theta = 48° 47'$$

To define its location, consider the sum of moments about point O. This is equivalent to the moment of resultant force about the same point, thus

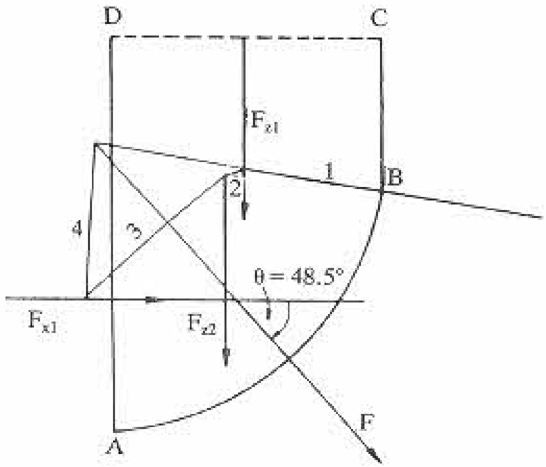
$$(12.5\gamma) \times (2.5 - 2.12) = (32.1\gamma) \times e$$
$$e = 0.148 \text{ m}$$

Also, we can solve the problem graphically by drawing the force and string polygons as illustrated below

$$Q = 8.5 \times (5\rho g) \times 5 = 212.5\,\rho g$$
$$= 2.08 \times 10^6 \quad N = 2.08\ MN$$
$$\theta = 48.5^\circ$$

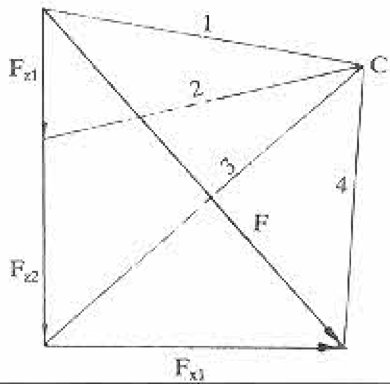
Figure scale
1 cm = 1 meter

String Polygon




Forces scale
1 cm = 5 × ρg N / m

Force Polygon

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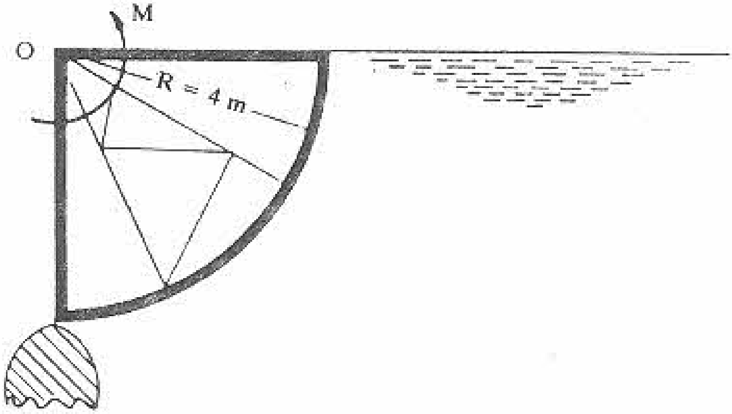
**EXAMPLE 2 – 8**

A tainter (radial) gate of $8 \times 10^5$N weight, 4 meter radius and 10 m width is pivoted at point 0 as shown in the Problem Description. A balancing moment of $2.4 \times 10^6$ N.m counter-clockwise is required to bring the gate to static equilibrium in the closing position. Find the reaction at the hinge and the line of action of the weight of the gate.

---

Problem Description

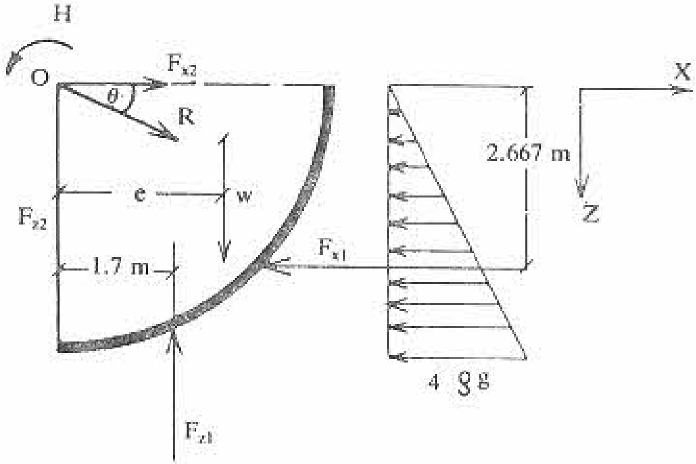



Data of the problem
* radius of the gate = 4 m
* width of the gate = 10 m
* weight of the gate = $8 \times 10^5$ N
* balancing moment, M = $2.4 \times 10^6$ N.m

Requirements
* reaction at the hinge for static equilibrium of the gate.
* the line of action of the weight of the gate for the same case.

Solution

The forces acting on the gate, are as shown in the figure. The lines of action of forces $F_{x1}$ and $F_{z1}$ are found by reference to Appendix A



The magnitude of forces $F_{x1}$, W and $F_1$ per one meter width of the gate are

$$F_{x1} = -\left(\frac{4\rho g}{2} \times 4\right) \times 1 = -8 \times \rho g = -78400 \text{ N/m} \qquad \text{(I)}$$

$$W = \frac{8 \times 10^5}{10} = 8 \times 10^4 \text{ N/m} \qquad \text{(II)}$$

$$F_{z1} = -\left(\frac{\pi \times 4^2}{4}\right) \times 1 \times \rho g = 12.57 \rho g = 123186 \text{ N/m} \qquad \text{(IIII)}$$

The force equilibrium equation of the gate in x-direction gives $\sum F_x = 0$, that is

$$F_{x2} = 78400 \text{ N/m} \qquad \text{(IV)}$$

Also, the force equilibrium in z direction $\sum F_z = 0$, that is

$$F_{z2} + 80000 - 123186 = 0$$

i.e.

$$F_{z2} = 43186 \text{ N/m} \qquad \text{(V)}$$

The reaction at the hing per one meter width R can be calculated from Eqs. (IV) and (V) as follows

$$R = \sqrt{F_{x2}^2 + F_{z2}^2} = \sqrt{(78400)^2 + (43186)^2}$$

$$= 89507.5 \text{ N/m} \qquad \text{(VI)}$$

The total reaction $R_T$ i.e. the reaction force for the 10m width becomes

$$R_T = 89507.5 \times 10 = 895075 \text{ N} \qquad \text{(VII)}$$

and its direction $\theta$ with respect to x-axis is given as follows

$$\tan\theta = \frac{F_{z2}}{F_{x2}} = \frac{43186}{78400} = 0.551$$

i.e.

$$\theta = 28^{\circ}\ 51'$$

The location of the line of action is determined by considering the sum of moments acting on the gate about a point such as 0. This summation equals zero at static equilibrium, therefore

$$240 - 80 \times e = 0$$
$$e = 3m$$

**EXAMPLE 2 – 9**

The drum gate shown in Fig. (I) has the dimensions given in the Problem Description with 20 meter width. The action line of the weight is located at 2.25m horizontally from the hinge. Find the weight of the gate. Also find the hinge reaction at the shown static equilibrium position.

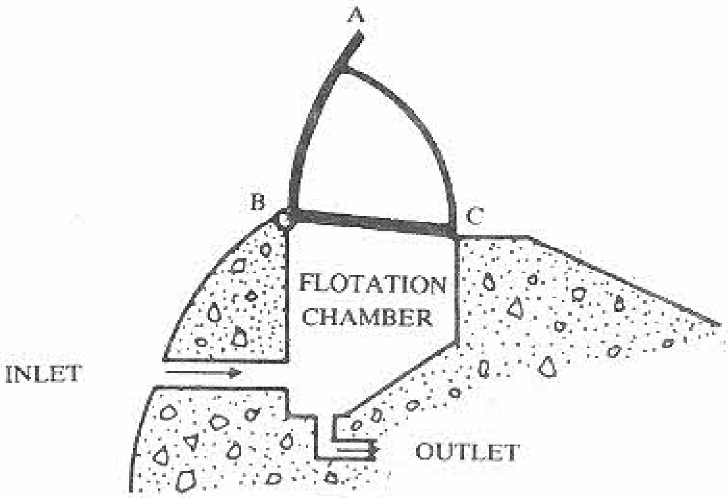
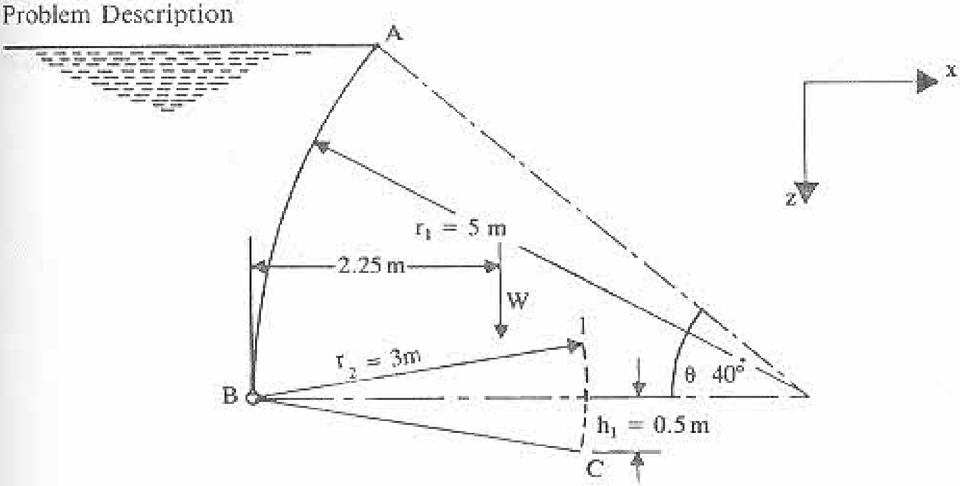


*Fig. I*

Problem Description



Data of the problem

* $r_1 = 5$ m
* $r_2 = 3$ m
* $h_1 = 0.5$ m
* $\theta = 40°$
* Action line of the weight is at a distance 2.25 m horizontally from the hinge.

Requirements

* weight of the gate, W
* hinge reaction, $(R,\theta)$

Solution

This example can be solved either analytically or graphically. The analytical solution is left as an exercise for the reader and only the graphical solution is considered here.

As a first step, calculate the resultants of the hydrostatic pressure per-one meter width in the x and z directions by using Appendix A.

$$F_{x1} = \frac{1}{2} \times (3.714\ \rho g) \times 3.714 = 6.9\ \rho g = 676200\ N/m$$

To get the hydrostatic force $F_{z1}$ in z-direction on AB (i.e. the force due to the weight of water in the volume having base area ABG and one meter width) consider the composite system resulting of adding the triangle ADE and the rectangle AEBG and subtracting the circular sector DAB. This gives

$F_{z1} = 1.188\ \rho g = 1164.4$ N/m

Also, $F_{z2}$ can be determined as follows

$$F_{z2} = \left(\frac{3.714\ \rho g + 3.214\ \rho g}{2}\right) \times 2.95 = 10.22\ \rho g$$
$$= 100156\ N/m$$

Locations of these forces are as shown in the following figure.

The second step is getting the requirements by drawing force and string polygons as shown in Fig. (II)

From the polygon of the forces, the weight of the gate is

W = 0.8624 MN

Also, the magnitude of hinge reaction R becomes

R = 1.6268 MN

with direction $\theta$ given as follows

$\theta = 213°\ 42'$

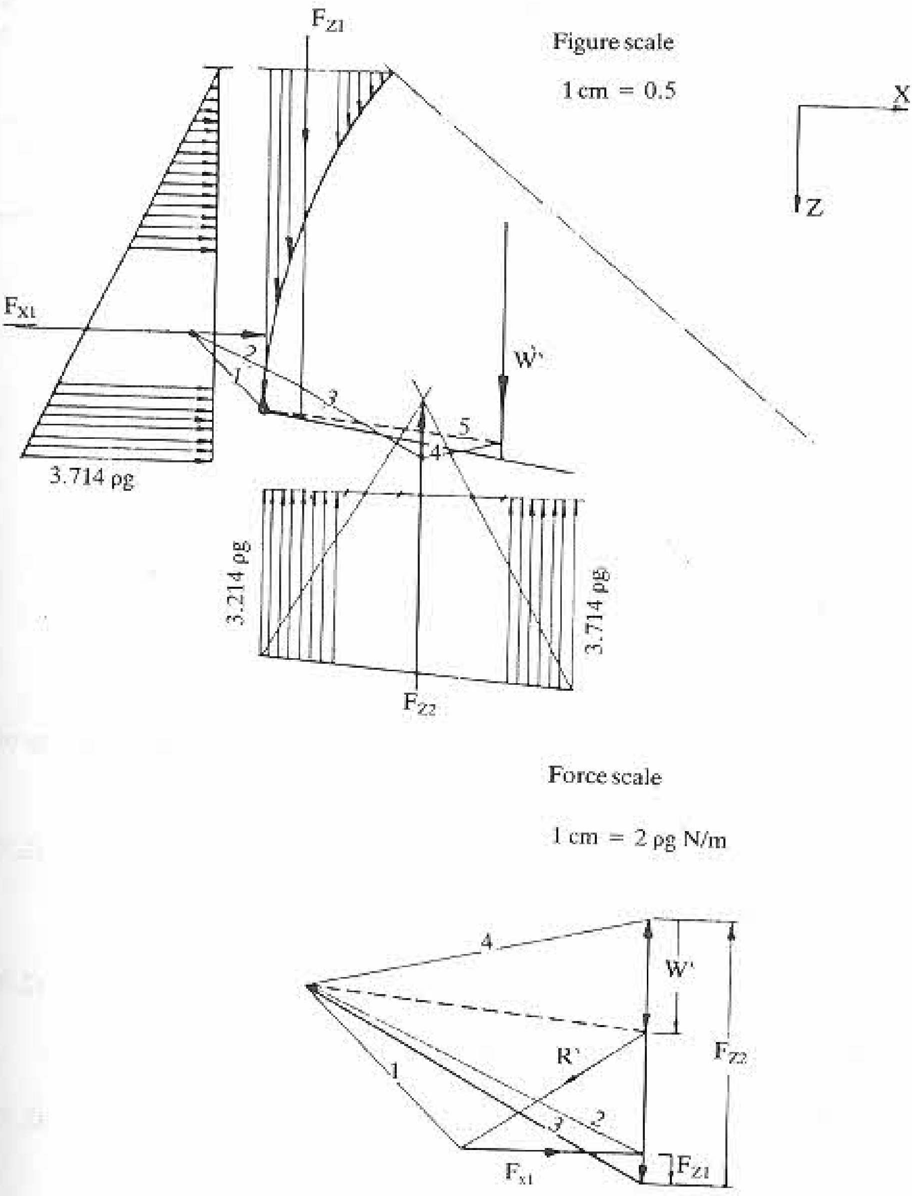


Figure scale
1 cm = 0.5
Force scale
1 cm = 2 ρg N/m
3.714 ρg
3.214 ρg
3.714 ρg
W'
R'


*Fig. II*

## 2 – 6 – Buoyancy

A body floating freely in an incompressible fluid, whether completely or partially immersed, is acted upon by two forces in static equilibrium. One is its weight and the other is the resultant force exerted on its surface by the surrounding fluid. The latter force is called the buoyant force.

To evaluate the buoyant force on a body totally immersed and floating freely in a fluid of density $\rho$, consider a vertical prism of cross-sectional area $dA_z$, taken

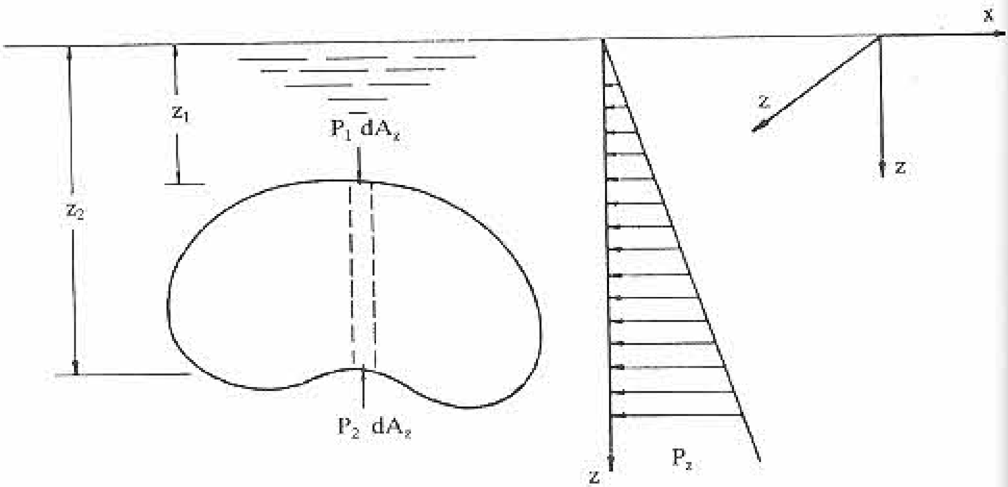



*Fig. 2.14 Submerged body*

from within the body as shown in Fig. (2.14). The pressure acting on the top of the prism is

$$P_1 = \rho g\ z_1 \tag{2.75}$$

whereas, the pressure on the bottom of the prism is

$$P_2 = \rho g\ z_2 \tag{2.76}$$

The net vertically upward force (upthrust) acting on the prism is

$$dF_B = \rho g\ (z_2 - z_1)\ dA_z \tag{2.77}$$

i.e.

$$dF_B = \rho g\ \ dV \tag{2.78}$$

where dV is the elemental volume of the prism and subscript B denotes buoyancy.

Integrating throughout the entire body, we get the following expression for the buoyant force

$$F_B = \int \rho g \, dV = \rho g V \tag{2.79}$$

This equation states that buoyant force on a floating completely submerged body in an incompressible fluid is equal to the weight of fluid displaced by the body.

The centre of buoyancy, i.e. the point of action of $F_B$, is determined by taking the moments about y and x axes respectively. This yields

$$F_B \cdot \bar{X}_B = \int_V x \rho g \, dV \tag{2.80a}$$

$$F_B \cdot \bar{Y}_B = \int_V y \rho g \, dV \tag{2.80b}$$

Substitution for $F_B$ from Eq.(2.79) gives

$$\bar{X}_B = \frac{\int_V x \, dV}{V} \tag{2.81a}$$

$$\bar{Y}_B = \frac{\int_V y \, dV}{V} \tag{2.81b}$$

It can be concluded from Eqs. (2.81) that the buoyant force acts through the centroid of the volume displaced by the body.

Equations (2.79) and (2.81) represent the principle of Archimedes (250.B.C), which states that: "When a body is completely or partially immersed in a fluid, it experiences an upthrust equal to the weight of the mass of that fluid displaced. This upthrust acts vertically through the centre of gavity of the displaced fluid".

In case of a partially immersed body (see Fig. 2.15) the same analysis can be applied to conclude that the total buoyant force is equal to the weights of both air

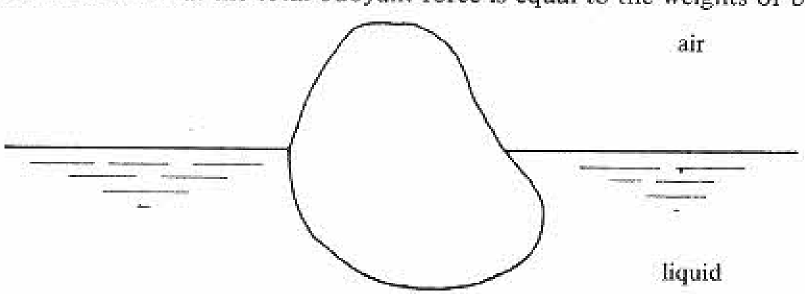



*Fig. 2.15 Emergent body*

and liquid displaced by the body. Since the density of air is negligible compared to that of the liquid, the buoyant force can be considered equal to the weight of the liquid displaced by the body and acts through the centroid of the displaced volume of the liquid.

## 2 – 7 – Static Stability of Floating Bodies

A floating body at rest is in static equilibrium under the two forces: body weight and buoyant force. These two forces must be equal but opposite and lie on the same line through the centre of gravity of the body, as shown in Figs. (2.16a) and (2.17a). The stability of the body is governed by the forces and moments produced as it is disturbed from the position of static equilibrium. The floating body is said to be stable if when it is subjected to a small disturbance, forces and moments will evolve to restore the body to its original position.

If the centre of gravity of a floating body (G) lies below its centre of buoyancy (B) the body will be stable. Figure (2.16) shows that a couple will be produced to restore the body to its original position. But if the centre of gravity of the floating body lies above its centre of buoyancy then either of two situations will evolve:

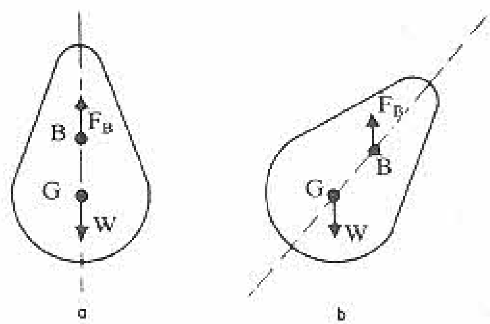



*Fig. 2.16 Stable equilibrium of a submerged body (G below B)*

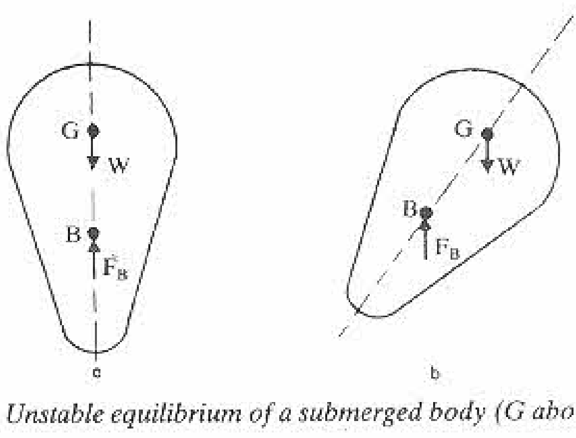



*Fig. 2.17 Unstable equilibrium of a submerged body (G above B)*

a) immersed floating body which will be unstable as illustrated by Fig. (2.17).

b) emergent floating body which may be stable even though its gravity centre is above its buoyancy centre. This will be explained in the following.

Consider a body partially immersed as shown in Fig. (2.18). If the body is subjected to a small angular displacement $\theta$ ($\tan\theta \cong \theta$ in radians) about the axis y, then the immersed part of the body is changed and the centre of buoyancy becomes $\grave{B}$ instead of B. To find $\grave{B}$, take the moment of the new buoyant force about B. The new buoyant force, $F_{\grave{B}}$, on the tipped configuration is equal to the original buoyant force $F_B$ at B, plus the buoyant force of the wedge YE$\grave{E}$ minus the buoyant force of the wedge YA$\grave{A}$. Since the original bouyant force, $F_B$, has zero moment about its centre and since both the buoyant forces of the wedge OE$\grave{E}$ and the wedge OA$\grave{A}$ make a couple. Then the balance of moment gives.

$$F_{\grave{B}} \cdot \bar{b} = \int_A x \cdot dF_{B,W} \qquad (2.82)$$

where $dF_{B,W}$ is the buoyant force of the infinitesimal volume ($x.\theta.dA$) of the wedges yE$\grave{E}$ and yA$\grave{A}$. It is obvious that the term $\int_A x \, dF_{B,W}$ represents the couple of the wedges. Substituting for $F_{\grave{B}}$ and $dF_{B,W}$ by ($\rho g V$) and ($\rho g \times \theta dA$) respectively, gives

$$V.\bar{b} = \theta \int_A x^2 \, dA \qquad (2.83)$$

or

$$\bar{b} = \frac{\theta . I_{yy}}{V} \qquad (2.84)$$

where

- $\rho$ is the density of the liquid,
- $V$ is the volume of the liquid displaced by the body,
- $dA$ is an infinitesimal area of the plane $\grave{A}$ $\grave{E}$,
- $I_{yy}$ is the second moment of the plane $\grave{A}$ $\grave{E}$ about the y axis ($I = \int_A x^2 \, dA$). The second moment of area of the plane AE can be used as I for small angular displacement.

By evaluating $\bar{b}$ the line of action of the new buoyant force $F_{\grave{B}}$ can be found. This line intersects the geometric centerline BG of the cross section at M. The point M is termed the metacentre and the distance between gravity centre and metacentre, GM, is termed the metacentric height. Clearly, the condition governing the stability of partially immersed floating bodies, in which the centre of gavity is above the buoyancy centre, depends on the position of the metacentre relative to the gravity centre. The body will be stable if its metacentre is above its gravity centre, neutral if they coincide and will be unstable if M is below G.

The metacentric height is of great importance to naval architects in

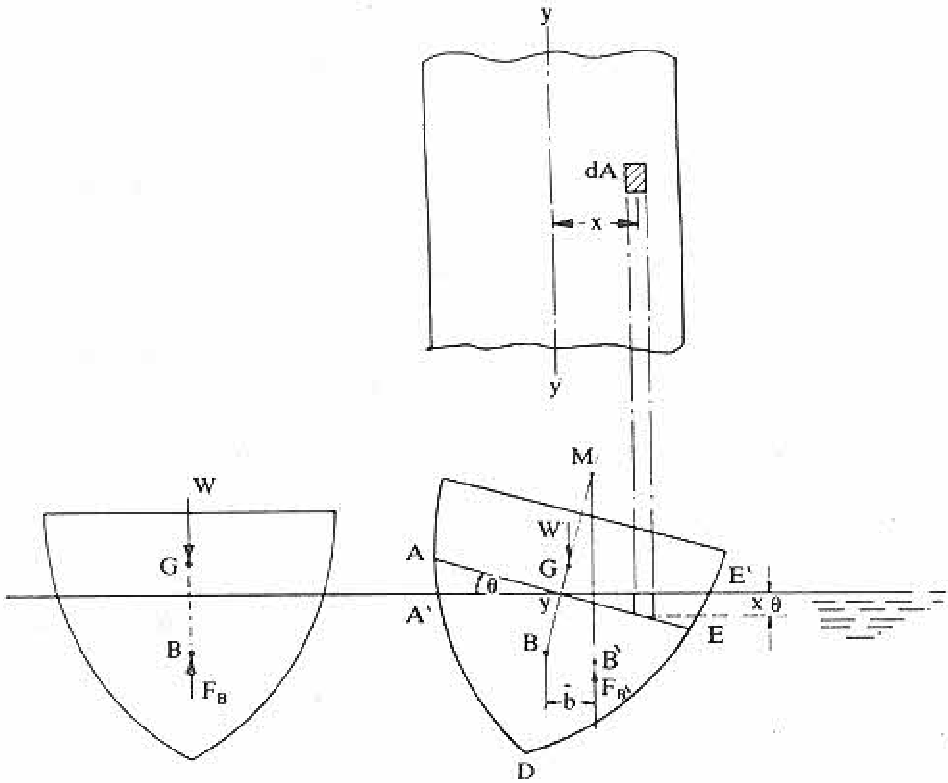
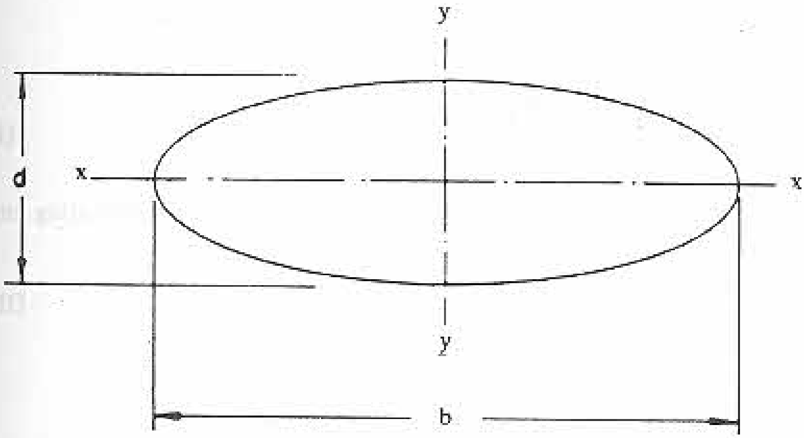



*Fig. 2.18 Stability of an emergent body*

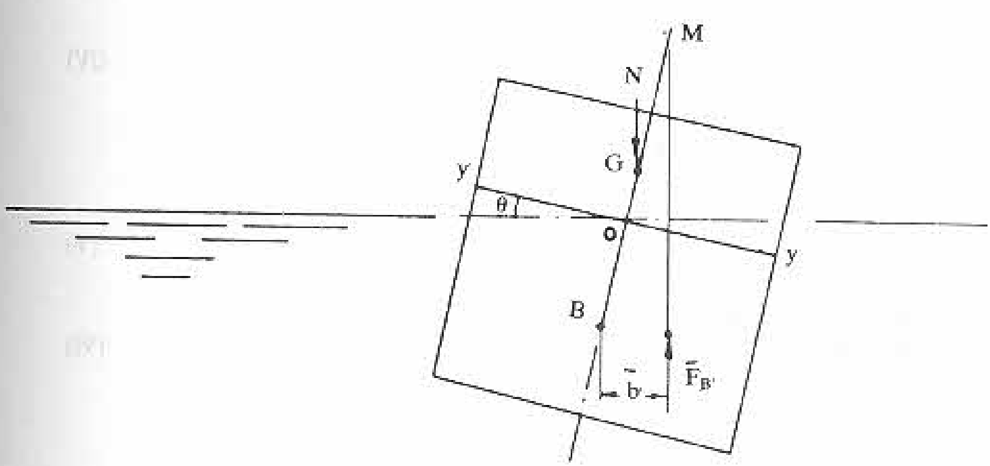
the design of ships. The degree of stability of the vessel increases with the increase of metacentric height, but on the other hand the period of roll also depends on the metacentric height. A too large value of metacentric height tends to result in undesirably rapid rolling, particularly in rough seas. Commercial vessels, particularly liners, are generally designed to have a relatively small metacentric height of the range 0.3 to 0.6 metres for rolling displacement about a longitudinal axis, whereas, the corresponding value for warships is between 0.6 to 2.0 meters, thus giving them a greater reserve of stability.

**EXAMPLE 2 – 10**

A barge of an ellipsoide cross-sectional area at the waterline as shown in the figure, where b and d are 45m and 15m respectively, weighs 2800 tons. It floats in salt water of density 1026 $kg/m^3$. The centre of gravity and the centre of buoyancy are 0.65m and 1.5m above and below the free surface respectively. Define the metacentric height about the longitudinal axis x-x.




Problem Description




Data of the Problem

* ellipsoide cross section of dimensions
  b = 45m on x-axis
  d = 15m on y-axis
* weight of the barge, W = 2800 tons
* density of the salt water, $\rho$ = 1026 kg/m$^3$
* OG = 0.65m above the free surface
* OB = 1.5 m below the free surface

Requirement

* define the metacentric height about the longitudinal axis x-x

Solution

From problem description we find that

$$\overline{GM} = \overline{BM} - \overline{GB}$$
$$= \frac{\bar{b}}{\sin\theta} - \overline{GB} = \frac{\theta . I_{xx}}{V.\sin\theta} - \overline{GB} \qquad \text{(I)}$$

Since $\sin\theta = \theta$ in radians for small angular rotation. Then, by substituting into Eq. (I) we get

$$\overline{GM} = \frac{I_{xx}}{V} - \overline{GB} \qquad \text{(II)}$$

Using Appendix B we obtain $I_{xx}$ as

$$I_{xx} = \frac{1}{64} \pi b d^3 = \frac{1}{64} \pi (45) (15)^3$$
$$= 7455.12 \quad m^4 \qquad \text{(III)}$$

Using the buoyancy law namely

$$\rho g V = W \qquad \text{(IV)}$$

we get

$1026 \times 9.8 \times V = 2800 \times 9.964 \times 10^3$

i.e.

$$V = 2774.7 \text{ m}^3 \qquad \text{(V)}$$

Using the data of the problem GB becomes

$$GB = OG + OB = 0.65 + 1.5 = 2.15 \qquad \text{(VI)}$$

By substituting from Eqs. (III), (V) and (VI) into (II), we get

$$GM = \frac{7455.12}{2774.7} - 2.15 = 0.537 \quad m$$

## PROBLEMS ON CHAPTER TWO

### Problems on Sections 2-1 to 2-3

2.1. What will be (a) the gage pressure, (b) the absolute pressure of water at a depth of 15m below the free surface. Assume the density of water to be 1000 $kg/m^3$ and the atmospheric pressure 101 $kN/m^2$.

2.2. A diver descends from the surface of the sea to a depth of 35m. What would be the pressure under which the diver would be working, assuming that the density of sea water is 1025 kg $m^{-3}$ and remains constant?

2.3. Determine the pressure in Pa at (a) a depth of 7m below the free surace of a body of water and (b) at a depth of 10m below the free surface of a body of oil of specific gravity 0.75.

2.4. What depth of oil, specific gravity = 0.85, will produce a pressure of 140 kN/$m^2$. What would be the corresponding depth of water.

2.5. What is the pressure in kN/$m^2$ absolute and gage at a point 4m below the free surface of a liquid having specific gravity of 1.53 if the atmospheric pressure is equivalent to 755mm of mercury. Consider the specific gravity of mercury as 13.6 and the density of water as 1000 kg/$m^3$.

2.6. Determine the depth of a tube filled with mercury (s = 13.6) if the gage pressure at the bottom is 230 kN/$m^2$.

2.7. What is the gage pressure and the absolute pressure of the gas in Fig. (2.19) if the barometric pressure is 770mm of mercury and the liquid is (a) water of density 1005 kg/$m^3$, (b) oil of specific weight 8000 N/$m^3$.

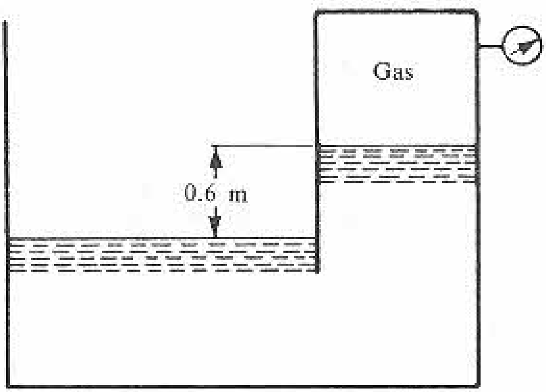



*Fig. 2.19*

2.8. An open tank contains oil of specific gravity 0.8 on top of water. If the depth of oil is 2.5m and the depth of water 4m, calculate the gage and absolute pressures at the bottom of the tank when the atmospheric pressure is 1 bar.

2.9. A hydraulic press has a diameter ratio between the two pistons of 9 to 1. The diameter of the larger piston is 65cm and it is required to support a mass of 4300kg. The press is filled with a hydraulic fluid of specific gravity 0.85. Calculate the force required on the smaller piston to provide the required force (a) when the two pistons are at same level, (b) when the smaller piston is 2.8m below the larger piston.

2.10. If the pressure gage shown in Fig. (2.20) indicates a pressure of $0.5 \times 10^5$ Pa, what are the pressure at points A, B, and C.

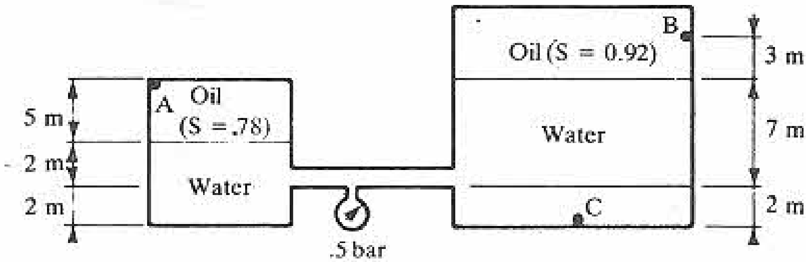



*Fig. 2.20*

2.11. A concrete dam is constructed as shown in Fig. (2.21). When the water level on the left is 13m, determine the pressure at the bottom and sketch the pressure distribution on the dam wall.

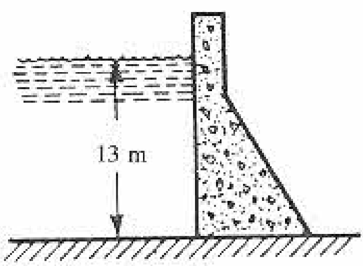



*Fig. 2.21*

2.12. Derive an expression for the pressure distribution in atmosphere from sea level to 20km by considering mean values for gravitation acceleration 9.79m/s$^2$, 9.759 m/s$^2$ for the troposphere and the stratosphere respectively. Sketch the pressure versus the altitude.

2.13. Assume that the temperature of the atmosphere diminishes with increasing altitude at the rate of 6.5°C per 1000m, find the pressure and density at a height of 8km if the corresponding values at sea level are 101 kN/m$^2$ and 1.235 kg/m$^3$ when the temperature is 15°C.

2.14. At an altitude z, of 11km, the atmospheric temperature T is −56.6°C and the pressure p, is 22.4 kN/m$^2$. Assuming that the temperature remains the same at higher altitudes, calculate the density of the air at an altitude of 16.5km. Assume R = 287 J/kg.K.

2.15. Calculate the pressure, temperature and density of the atmsphere at an altitude of 1500m if at zero altitude the temperature is 15°C and the pressure 101 kN/m$^2$. Consider $\gamma$ = 1.4, and R = 287 J/kg.K.

2.16. The pressure and the temperature at sea level are 101.32kPa and 20°C. If the troposphere is 11.5km high, find the atmospheric pressure, temperature, and density at 16km altitude.

2.17. Assuming that atmospheric temperature decreases with increasing altitude at a uniform rate of 0.006 K/m, determine the atmospheric pressure at an altitude of 8000 m if the temperature and pressure at sea level are 15°C and 10 N/cm$^2$ respectively.

2.18. At the top of a mountain the temperature is −5°C and a mercury barometer reads 56.6cm, whereas the reading at the foot of the mountain is 74.9cm. Assuming that R of atmospheric air is equal to 287 J/kg K and that temperature relates with pressure in the from $PT^{\gamma/(1-\gamma)}$ = constant, where $\gamma$ = 1.4, Find the height of the mountain assuming that gravitational acceleration is constant.

**Problems on Section 2-4**

2.19. Find the pressure in N/m$^2$ at point A for each case of Fig. (2.22). All dimensions in cms

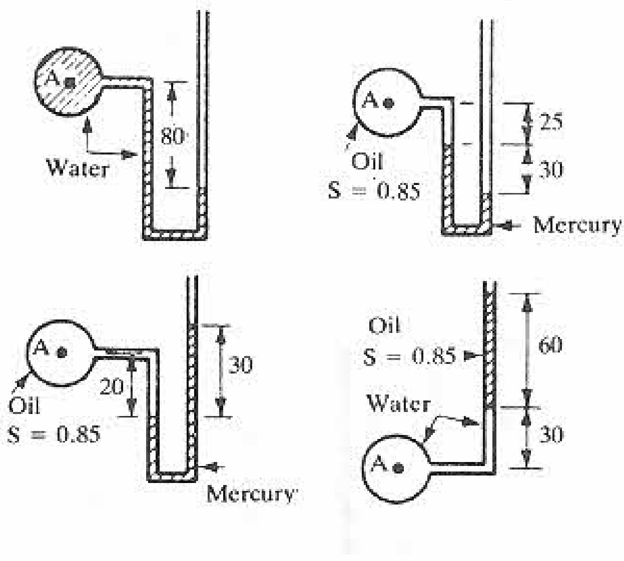



*Fig. 2.22*

2.20. Determine the difference in pressure between points A and B in separate pipes connected to a differential manometer as shown in Fig. (2.23). The basic manometer fluid is mercury. All dimensions in cms.

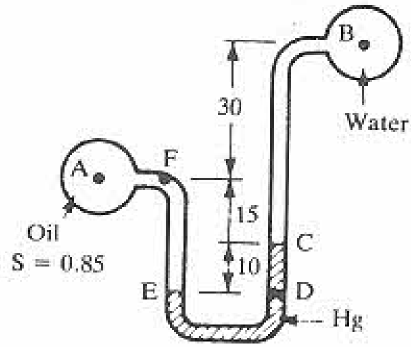



*Fig. 2.23*

2.21. What is the difference in water pressure between points A and B in Fig. (2.24) if the fluid at the top of the manometer is (a) oil (s = 0.93), (b) air.

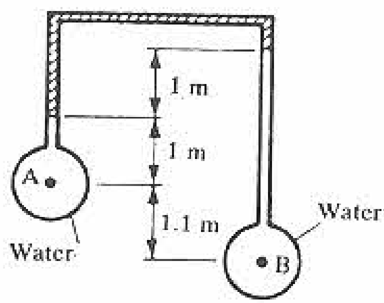



*Fig. 2.24*

2.22. In Fig. (2.25), if the water pressure at point B is 178.5kPa, what is the water pressure at point A? (All dimensions are in cms).

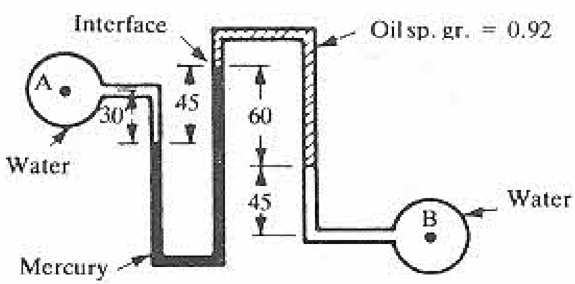



*Fig. 2.25*

2.23. In Fig. (2.26), if a = 5m, to what height will the oil rise in the oil column b?

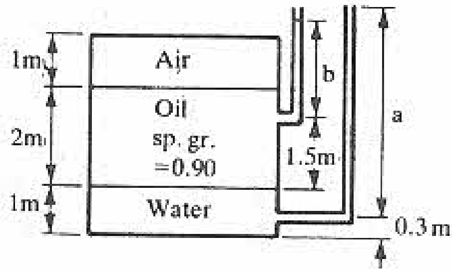



*Fig. 2.26*

2.24. Compute the atmospheric pressure on a day when the height of the mercury barometer is 743mm.

2.25. What would be the pressure in $kN/m^2$ if the equivalent head is measured as 30cm of (a) mercury of specific gravity 13.6, (b) water, (c) oil of specific weight 7.9 $kN/m^3$, (d) a liquid of density 640 $kg/m^3$.

2.26. A pump is a device that puts energy into a liquid in the form of pressure. The inlet side of a pump usually operates at less than atmospheric pressure as shown in Fig. (2.27). A manometer and a vacuum gauge are connected to the inlet side and the vacuum gauge reads the equivalent of 32.4 kPa absolute.

a) Express the gauge reading in kPa gage.

b) Calculate the deflection when the manometer liquid is mercury.

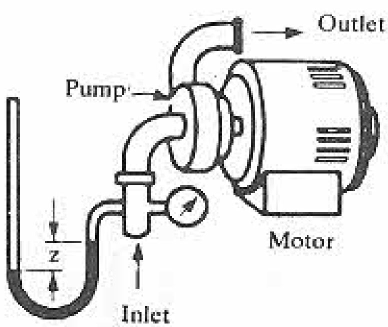



*Fig. 2.27*

2.27. Calculate the air pressure in the tank shown in Fig. (2.28). Take atmospheric pressure to be 101.25 kPa.

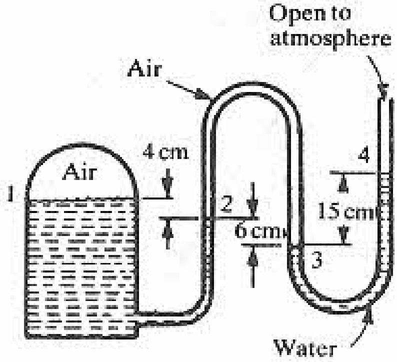



*Fig. 2.28*

2.28. For the sketch of Fig. (2.29), determine the pressure of the linseed oil if the glycerine pressure is $10^6 N/m^2$. All dimensions are in meters.

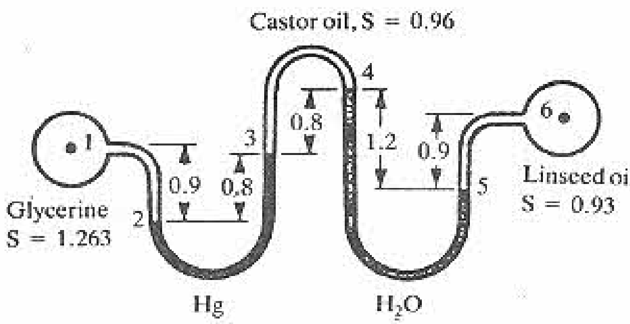



*Fig. 2.29*

2.29. For the W-tube water filled manometer configuration and the readings as shown in Fig. (2.30), calculate the absolute pressures at A and B. Take the atmospheric pressure 101.12 $kN/m^2$. All dimensions are in cms.

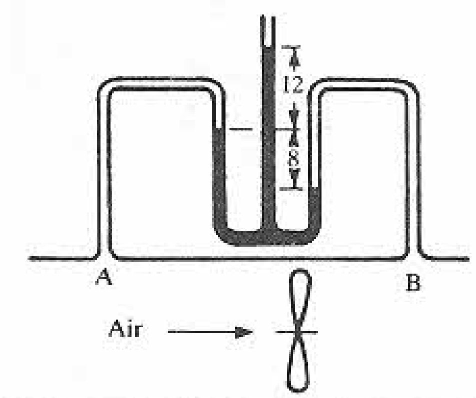



*Fig. 2.30*

2.30. Two U-tube manometers, one upright and the other inverted type, are connected across a water line and an oil line as shown in Fig. (2.31). If $h_1 = 6.4cm$, what shall be $h_2$ ?

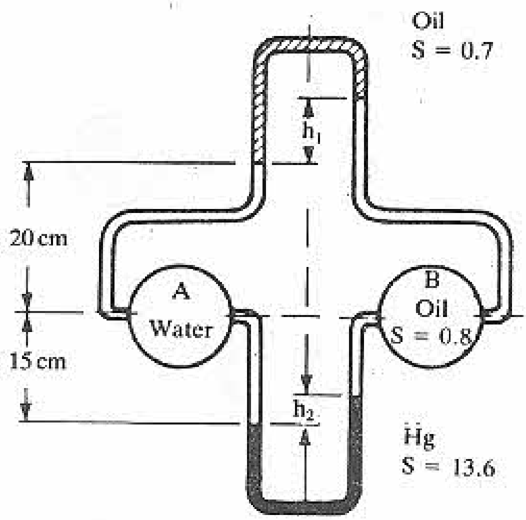



*Fig. 2.31*

2.31. A differential manometer is used to measure the pressure rise across a water pump. The liquid in the manometer is mercury with 13.6 specific gravity. The observed manometer deflection is 75cm and the manometer leads are connected to the pump as shown in Fig. (2.32).What is the pressure rise $P_1 - P_2$ in Pa.

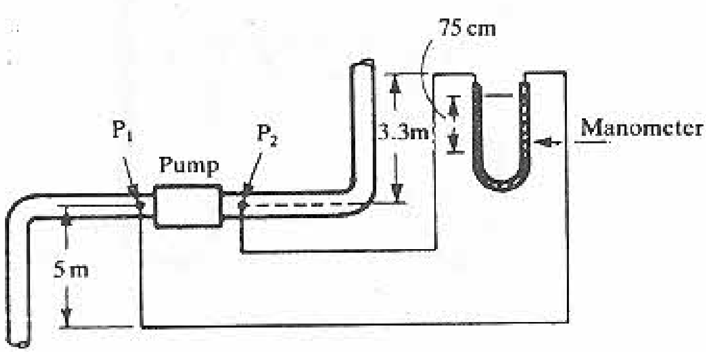



*Fig. 2.32*

2.32. Flasks (1) and (2) are connected through a differential manometer as shown in Fig. (2.33). What is the pressure difference between A and B if Fluid 1 is oil with specific gravity 0.86, Fluid 2 is mercury, and Fluid 3 is water.

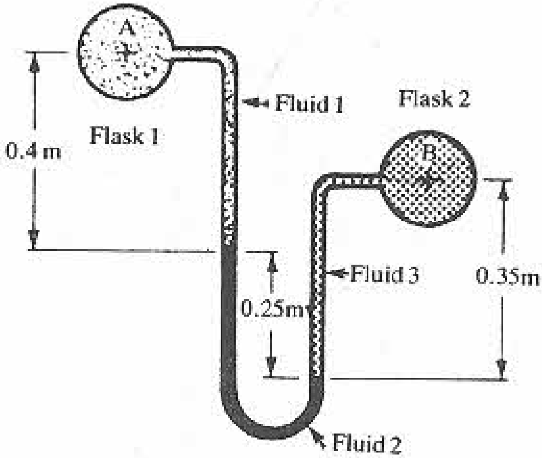



*Fig. 2.33*

2.33. What is the gauge pressure at point A for the system shown in Fig. (2.34). All dimensions are in cms.

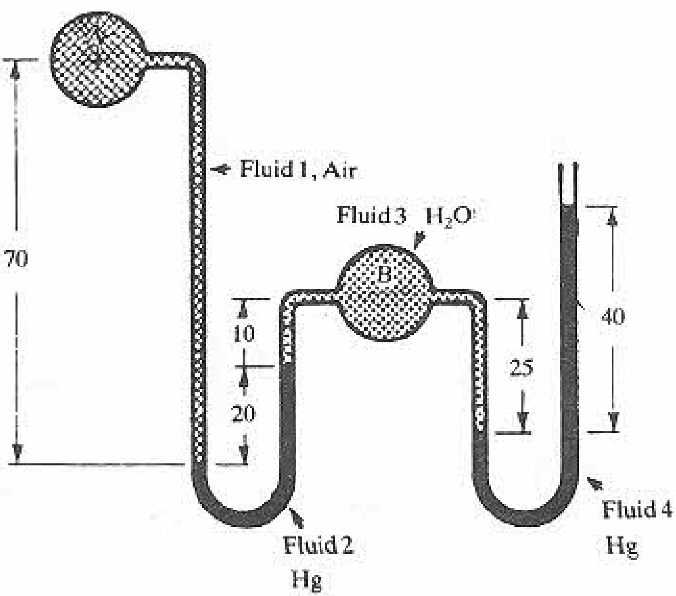



*Fig. 2.34*

2.34. Determine the difference in pressure between the points A and B for flow in the vertical pipe of Fig. (2.35).

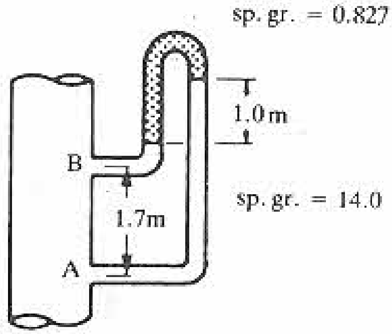



*Fig. 2.35*

2.35. An inclined manometer is used to measure the difference in air pressure in a pipe between two points as shown in Fig. (2.36). Find the difference in pressure for the given conditions. All dimensions are in cms.

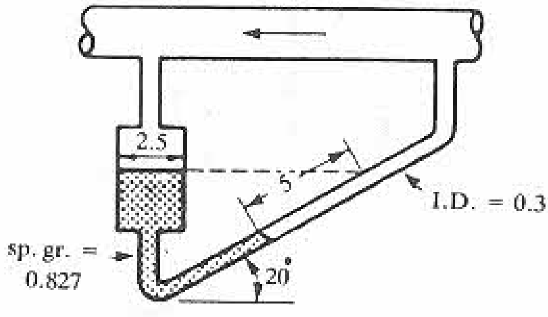



*Fig. 2.36*

2.36. The sensitive differential manometer shown in Fig. (2.37) has two liquids of density $\rho_1$ and $\rho_2$. Find an expression for the pressure difference $P_A - P_B$ in terms of $\rho_1$, $\rho_2$, z, $d_1$, and $d_2$.

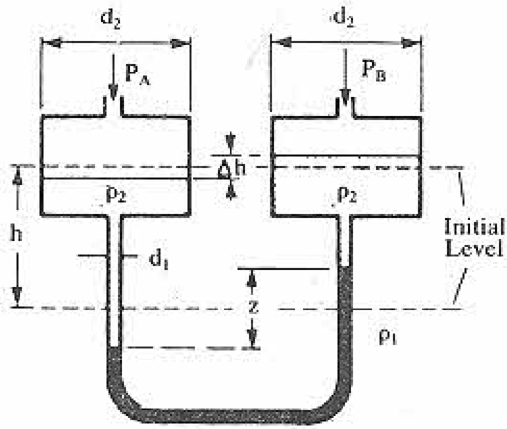



*Fig. 2.37*

**Problems on Section 2-5**

2.37. A gate ABC, 4m wide holds a 2m high stationary column of water as shown in Fig. (2.38). Calculate the tension in the string SA and the reaction at the hinge C.

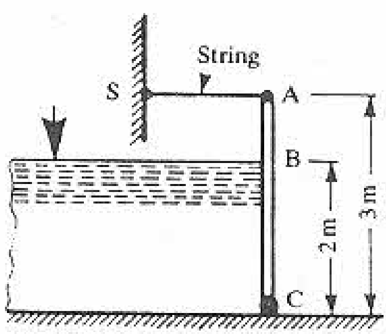



*Fig. 2.38*

2.38. A 1.5m diameter gate AB closes the side of a water tank as shown in Fig. (2.39). Determine the magnitude and location of force F to hold the gate without causing a reaction at the hinge at A.

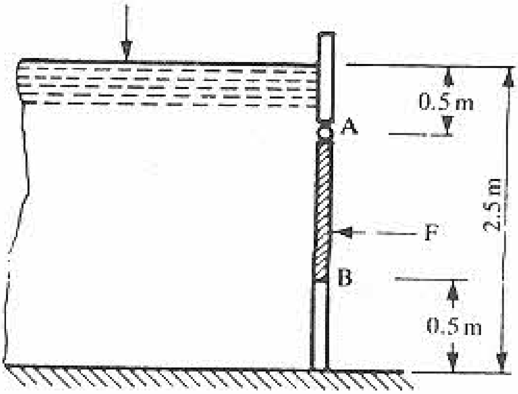



*Fig. 2.39*

2.39. Determine the height of liquid of specific gravity 0.8, to tip the rectangular flash board shown in Fig. (2.40).

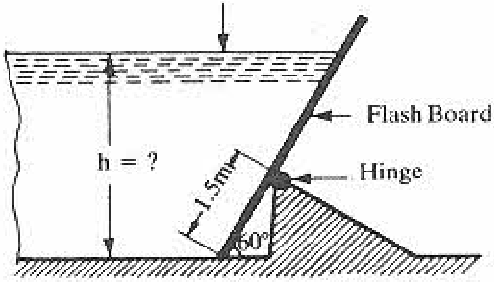



*Fig. 2.40*

2.40. A 1m x 2m cover AB on a container under a pressure as shown in Fig.(2.41) is held in position by a force F. Calculate the force and the reaction at the hinge B.

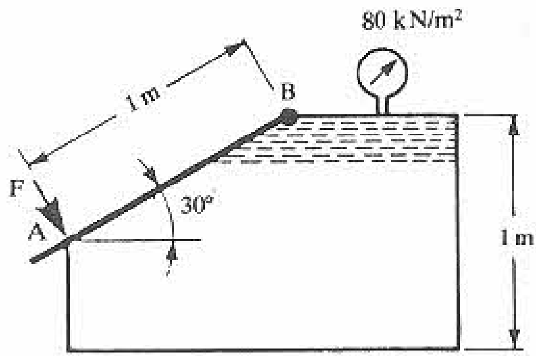



*Fig. 2.41*

2.41. As water rises on the left side of the rectangular gate in Fig. (2.42) it will open automatically. At what depth above the hinge will this occur? Ignore the weight of the gate.

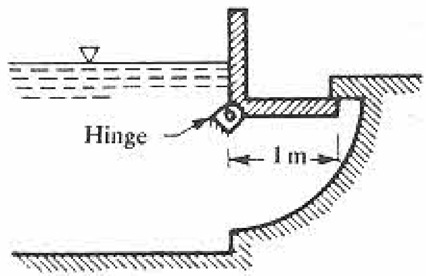



*Fig. 2.42*

2.42. A plate of weight 280 N per unit width is suspended at one end by a hinge at the water level of the reservoir as shown in Fig. (2.43). The bottom end is free to move. Calculate the angle of repose, θ, of the plate.

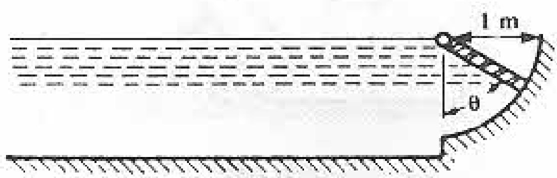



*Fig. 2.43*

2.43. Calculate the magnitude, direction and location of the horizontal and vertical componenets of the force experienced by each of the curved surfaces AB enclosing water as shown in Fig. (2.44).

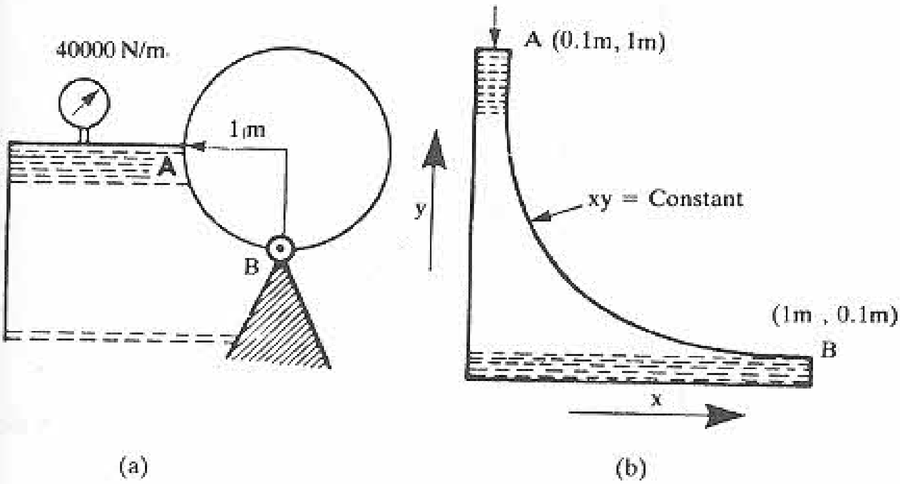



*Fig. 2.44*

2.44. A vertical sluice gate is 3.5m x 10m submerged in water and has 1m opening as in Fig. (2.45). From the pressure measurements reproduced below, calculate the horizontal force on the sluice gate

| z | 0 | 1 | 2 | 2.25 | 2.5 | 2.75 | 3.0 | 3.25 | 3.5 m |
|---|---|---|---|---|---|---|---|---|---|
| $P_g$ | 0 | 9.8 | 19 | 21 | 23 | 26 | 20 | 15 | 0 kN/m$^2$ |

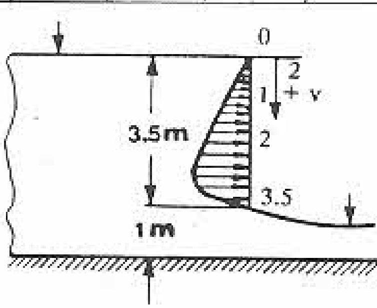



*Fig. 2.45*

2.45. Refering to Fig. (2.46), calculate the total force acting on one side of the plate ABDC, due to water at rest.

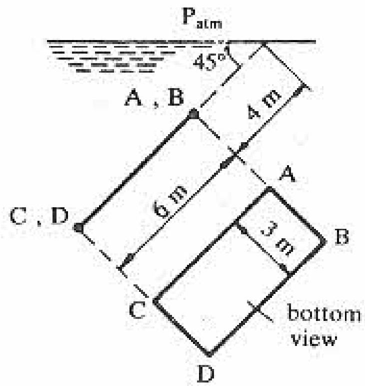



*Fig. 2.46*

2.46. The gate shown in Fig. (2.47) is 5m wide perpendicular to the paper. Calculate the total force on the gate due to oil of 0.88 specific gravity.

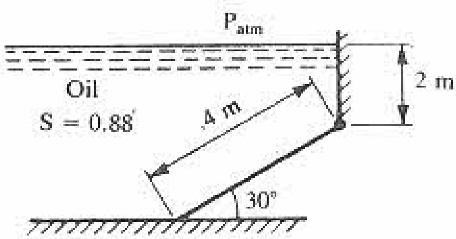



*Fig. 2.47*

2.47. A rectangular water gate as shown in Fig. (2.48) is 4m wide (perpendicular to the paper). What is the torque about hinge 0 due to the hydrostatic force.

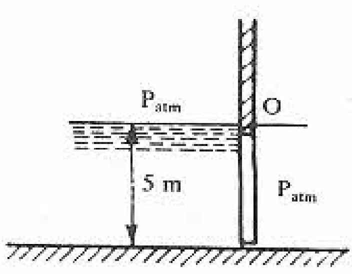



*Fig. 2.48*

2.48. Calculate the hydrostatic force acting on the vertical wall AB as shown in Fig. (2.49). Also determine the point of action of this force.

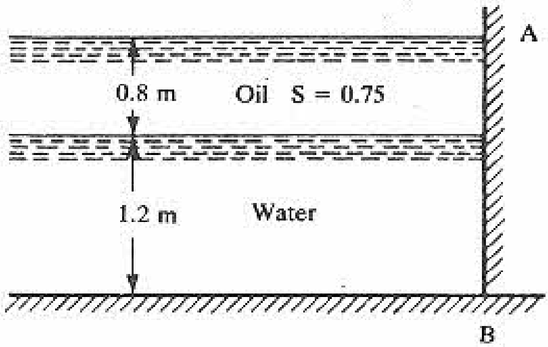



*Fig. 2.49*

2.49. Calculate the force per unit width on the wall OA of Fig. (2.50).

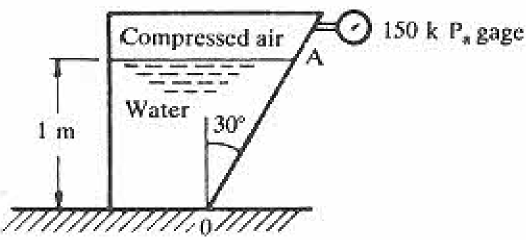



*Fig. 2.50*

2.50. A vertical water gate 2.4m tall and 1.2m wide is exposed to the atmosphere on one side. Its other side is in contact with water at rest to a dept of 1.9m; the remaining 0.5m of its height is above the water surface, and is exposed to the atmosphere. Find the force required to hold the gate up.

2.51. Calculate the minimum vertical force F required to keep the cover of this box, Fig. (2.51), closed. The cover is 3m wide perpendicular to the plane of the paper.

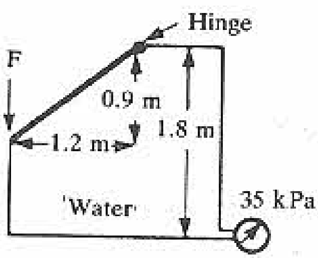



*Fig. 2.51*

2.52. Calculate magnitude and location of the total force on one side of this vertical plane area, Fig. (2.52).

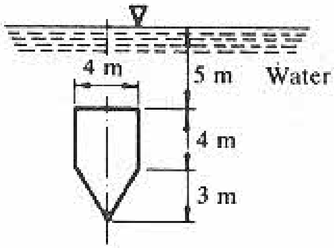



*Fig. 2.52*

2.53. What depth of water will cause this rectangular gate to fall? Neglect the weight of the gate.

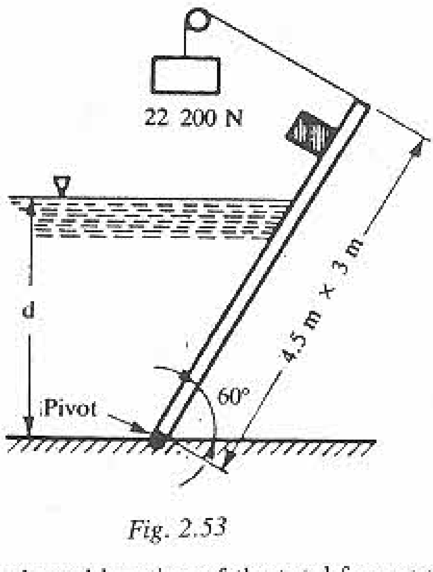



*Fig. 2.53*

2.54. Calculate magnitude and location of the total force on one side of this vertical plane area.

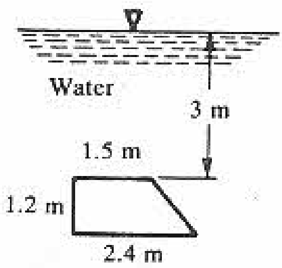



*Fig. 2.54*

2.55. Calculate the resultant hydrostatic force of water acting on 10m width of the wall ABCD shown in Fig. (2.55).

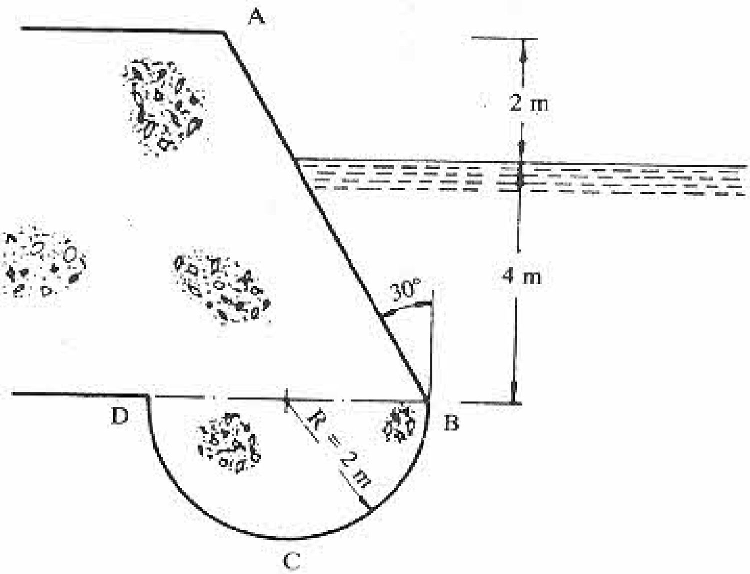



*Fig. 2.55*

2.56. Calculate the resultant hydrostatic force of water acting on 5m width of the wall ABC shown in Fig. (2.56).

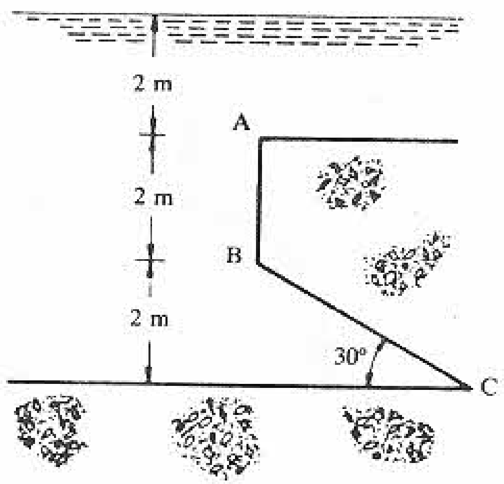



*Fig. 2.56*

2.57. Calculate the resultant hydrostatic force of water acting on 10m width of the wall ABC shown in Fig. (2.57).

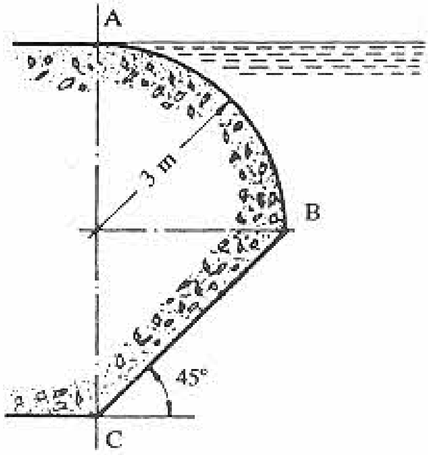



*Fig. 2.57*

2.58. In Fig. (2.58) the gate AB is a quarter-circle 2.5m wide, hinged at B. Find the force F at A to keep the gate closed.

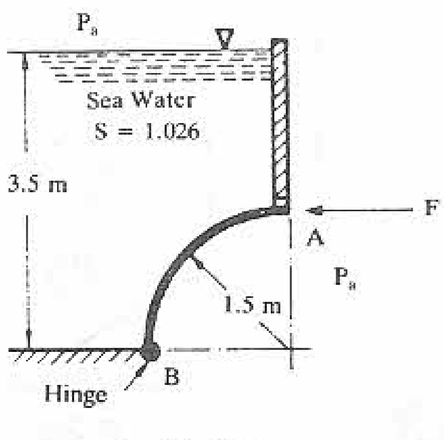



*Fig. 2.58*

2.59. A valve is located in a square tube which is connected with a reservoir as shown in Fig. (2.59). Find the resultant of the hydrostatic forces and the turning moment required to hold the valve in vertical position.

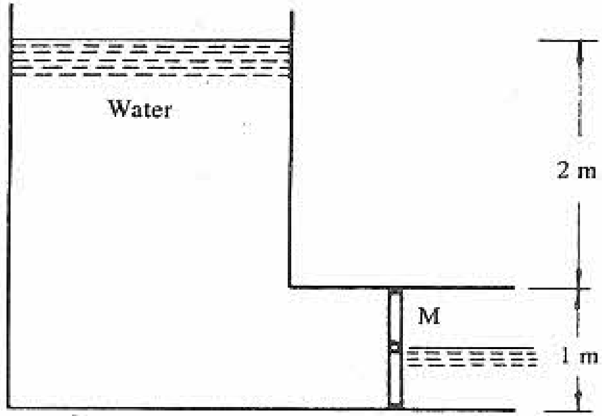



*Fig. 2.59*

2.60. Fig. (2.60) shows a sector water gate. The width of the gate perpendicular to the paper is 5m. Find the hydrostatic force exerted on the gate by the dammed water and the torque about the axle O required to hold the gate in place.

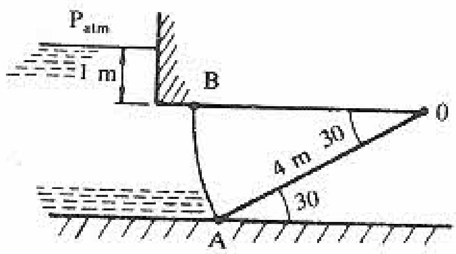



*Fig. 2.60*

2.61. Water is dammed up by a circular cylinder floating flush with the water surface as shown in Fig. (2.61). The length of the cylinder in the direction perpendicular to the paper is 3m. Find the weight of the cylinder and the horizontal force acting on the wall by the cylinder.

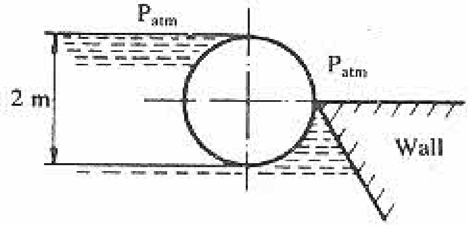



*Fig. 2.61*

2.62. An inclined triangular gate is shown in Fig. (2.62). Water on its top side fills up to the hinge 0. Its back side is exposed to atmospheric pressure. Find the hydrostatic force acting on the gate.

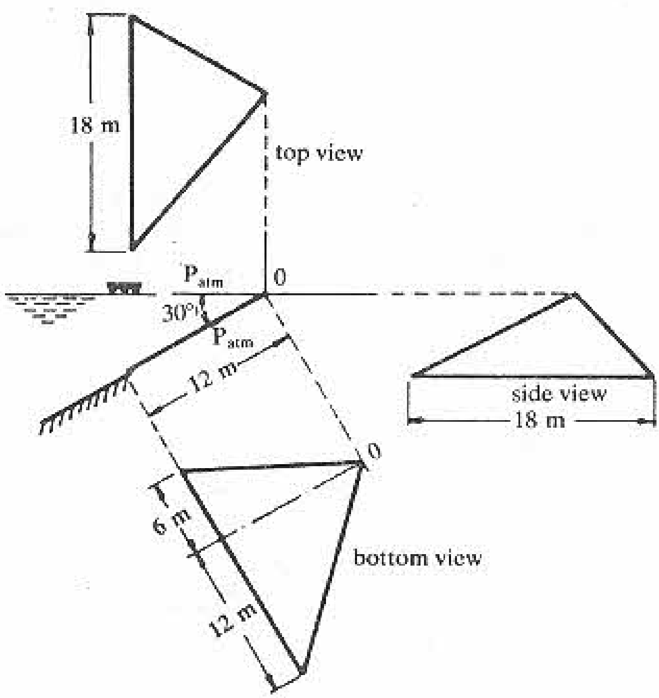



*Fig. 2.62*

2.63. An engineer has designed a pressure cooker as shown in Fig. (2.63). On top of the cooker, a tube with 6mm inside diameter is to be used as the valve port. Steel balls (with density 7800 kg/m$^3$) are to be placed on the tube opening as shown. Calculate the radii needed for pressure regulated at 70, 105, and 140 kPa gauge inside the cooker.

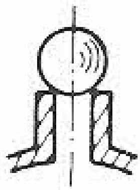


*Fig. 2.63*

2.64. Determine the vertical and horizontal components of the hydrostatic force per unit width acting on the hinged gate shown in Fig. (2.64). Would the water tend to rotate the gate?

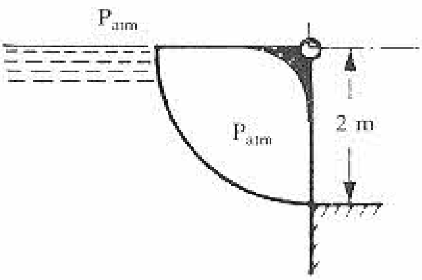



*Fig. 2.64*

2.65. Repeat problem 2.64 with the water surface lowered so that the line of contact of the water surface with the gate centre is 15° below the horizontal.

2.66. Calculate the hydrostatic force, as well as the torque of this force about the axle 0, exerted by the water at rest on the gate as shown in Fig. (2.65).

width = 5 m
perpendicular
to paper

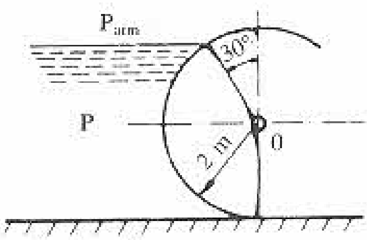



*Fig. 2.65*

2.67. The gate shown in Fig. (2.66) is an integral piece 4m wide (in the direction perpendicular to the paper) and is hinged at 0. A vertical force F is required to hold the gate. Neglecting the weight of the gate, find F.

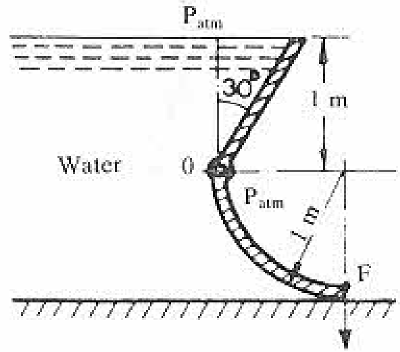



*Fig. 2.66*

2.68. Find the hydrostatic force per unit width on the iron block which is shown in Fig. (2.67).

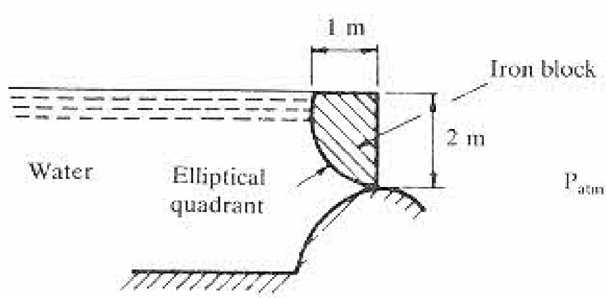



*Fig. 2.67*

**Problems on Sections 2-6 and 2-7**

2.69. Calculate the diameter of the spherical float, 0.15kg, so as to lift the circular plug vlave which opens as soon as the water level reaches 1.8m above the plug. The mass of the plug is 5kg and the length of the string is 1.8m, see Fig. (2.68).

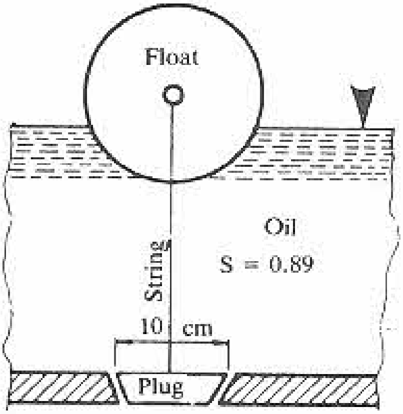



*Fig. 2.68*

2.70. A 3-kg mass of wrought iron, specific gravity = 7.8, floats in a beaker of mercury. What fraction of the iron is submerged? If sufficient water is poured into the beaker to cover the iron completely, what fraction of the iron is submerged in the mercury?

2.71. A hydrometer, see Fig. (2.69), has a mass of 0.8kg. The graduated stem BA is 0.25m long and has a cross-sectional area of 0.5 $cm^2$, while the bulb below B has a volume of 107.3 $cm^3$. What fraction of the stem BA will be submerged when the hydrometer floats in water? What is the lowest density the hydrometer can read?

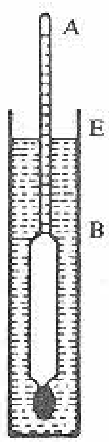



*Fig. 2.69*

2.72. A rectangular block of wood has a cross-sectional area of 120 $cm^2$ and a height of 0.5m. It floats vertically in water because to its lower edge is fastened a weight of 0.6 kg of lead. How much of the wood projects out of the water? (Take the density of wood to be 562 $kg/m^3$).

2.73. A solid circular cylinder of radius r and height h has specific gravity 0.72. Find the minimum ratio r/h for which the cylinder will float in water with its axis vertical and just be stable.

2.74. An ocean liner 250m long, 30m wide displaces 65 MN of water. The second moment of inertia of the water plane about its fore-and aft axis is 65% of the circumscribing rectangle. The position of the centre of buoyancy is 2.65 m below the centre of gravity. Locate the metacentre.

2.75. A block of wood, 0.2m x 0.5m in cross section and 0.8m long has a mass of 60kg. Can the block float with the 0.5m side vertical.

2.76. A rectangular pontoon floating in sea water, $\rho = 1032$ $kg/m^3$, is 25m long, 7.5m wide, 2.5m deep and weighs 150 tons. It carries on its upper deck a load of 100 tons. The centre of gravity of the load is 2.5m above the deck and that of the pontoon is 1.2m below the deck. Find the metacentric height.

2.77. A buoy, as shown, consists of a wooden pole 30cm in diameter and 1.7m long, with a semispherical weight at the bottom. The specific weight of the wood is 0.63 and of the bottom 7.6. (a) Find the positions of the center of gravity and the center of buoyancy from the top of the buoy. (b) Find the metacentric height.

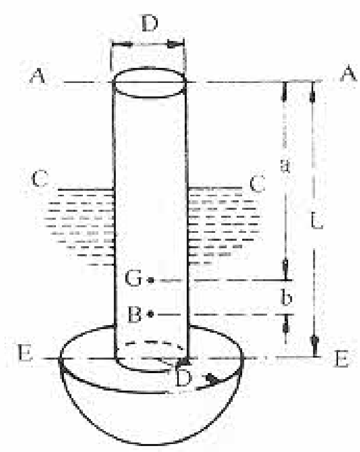



*Fig. 2.70*

2.78. A buoy, floating in sea water of density 1030 $kg/m^3$ is concial in shape with a diameter across the top of 1.4m and a vertex angle of 60°. Its mass is 300kg and its centre of gravity is 750mm from the vertex. A flashing beacon is to be fitted to the top of the buoy. If this unit is of mass 55kg what is the maximum height of its centre of gravity above the top of the buoy if the whole assembly is not to be unstable? (The centre of volume of a cone of height h is at 3h/4 from the vertex.)

2.79. A rectangular pontoon 11m long, 7.2m broad and 2.5m deep has a mass of 70000kg. It carries on its upper deck a horizontal boiler of 4.8m diameter and a mass 50000kg. The centers of gravity of the boiler and the pontoon may be assumed to be at their centres of figure and in the same vertical line. Find the metacentric height. Density of sea water is 1025 $kg/m^3$.

2.80. A rectangular pontoon has a mass of 245 metric tons and a length of 20m. The centre of gravity is 0.3m above the centre of cross-section and the metacentric height is to be 1.2m when the angle of heel is 10 deg. The freeboard must not be less than 0.6m when the pontoon is vertical. Find the breadth and height of the pontoon if floating in fresh water.

2.81. A buoy carries a light and has a cylindrical upper portion of 2.2m diam and 1.3m deep. The lower portion which is curved displaces a volume of 0.396 $m^3$ and its centre of buoyancy is situated 1.28m below the top of the cylinder. The centre of gravity is situated 0.9m below the top of the cylinder and the total displacement is 2.6 metric tons. Find the metacentric height. Density of sea water is 1025 $kg/m^3$.

CHAPTER THREE

# BASIC RELATIONS GOVERNING THE BEHAVIOUR OF NON-VISCOUS FLOWS

## 3 – 1 – Introduction

In the present chapter basic relations that govern the behaviour of non-viscous flows are considered. The assumption of non-viscous flow is verified in many practical applications. As it will be shown in the next chapter the effect of viscosity of the fluid is appreciable within a thin layer known as the boundary layer outside of which the flow is practically non-viscous.

In the following the conservations of mass and momentum are presented and applied for both finite and infinitesimal control volumes. In addition the conservation of energy when restricted to special flow conditions leads to what is known as Bernoulli's equation. Useful applications of these basic relations are given with illustrations.

## 3 – 2 – Velocity and Acceleration in a Fluid Continuum

A fluid element in a fluid continuum may have velocity components u, v and w in direction x, y and z respectively. Each of u, v and w may be a function of the independent variables, x,y,z and t, so that

$$\left.\begin{aligned} u &= u\ (x,\ y,\ z,\ t) \\ v &= v\ (x,\ y,\ z,\ t) \\ w &= w\ (x,\ y,\ z,\ t) \end{aligned}\right\} \qquad (3.1)$$

At time (t + dt), the position of the fluid element becomes (x + dx, y + dy, z + dz). Refering to Appendix E the total changes in the velocity components are given as follows

$$\left.\begin{aligned} du &= \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \\ dv &= \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz \\ dw &= \frac{\partial w}{\partial t}dt + \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz \end{aligned}\right\} \qquad (3.2)$$

Acceleration of the fluid element in any of the directions x, y and z is the rate of change of velocity component in that direction with respect to time. For example: the acceleration in the x direction $a_x = du/dt$ so that at

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} \qquad (3.3a)$$

But since the velocity components u, v and w are equal to $\frac{dx}{dt}, \frac{dy}{dt}$ and $\frac{dz}{dt}$ respectively, then Eq. (3.3a) becomes

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \qquad (3.3b)$$

Similarly, the acceleration in the directions y and z take the forms

$$a_y = \frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \qquad (3.3c)$$

$$a_z = \frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \qquad (3.3d)$$

## 3–3– Rotational and Irrotational Flow

A fluid element is said to have zero rotation in a plane if the average of the angular velocities of any two of its mutually perpendicular lines in that plane is zero. For example, if one line rotates in an anticlockwise direction at the same rate as the other rotates in a clockwise direction the particle is distorting but not rotating.

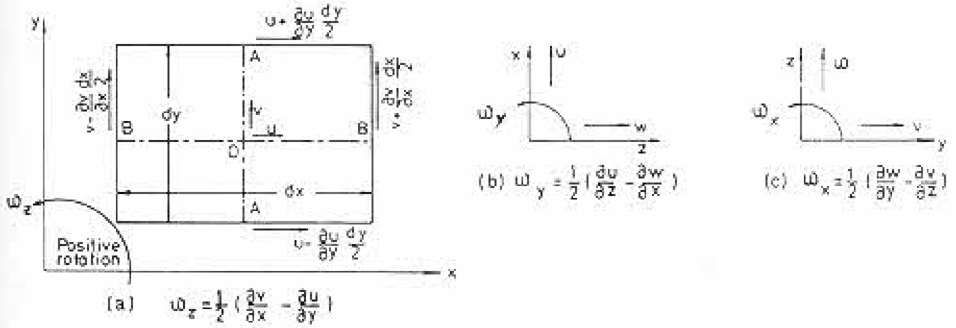



*Fig. 3.1 Rotation of a fluid element*

In Fig. (3.1) consider a rectangular element in two dimensional flow of which two mutually perpindicular lines are AA and BB. The velocity components at the centre of the element are u and v. The values of u and v along the lines AA and BB change with the coordinates x and y as shown in the figure. In a left-hand coordinate system, positive rotation around axis z is the anticlockwise rotation, then the rotation of line AA is given as follows

$\omega_{OA}$ = relative velocity between any two points on the line/distance between the two points

$$= -\frac{\partial u}{\partial y}\frac{dy}{2} \Big/ \frac{dy}{2} = -\frac{\partial u}{\partial y}$$

and rotation of line BB may be given as follows

$$\omega_{OB} = \frac{\partial v}{\partial x}\frac{dx}{2} \Big/ \frac{dx}{2} = \frac{\partial v}{\partial x}$$

The average rotation of these two lines would then be

$$\omega_z = \tfrac{1}{2}(\omega_{OA} + \omega_{OB}) = \tfrac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad (3.4a)$$

Similarly it can be shown that, rotation about axis y

$$\omega_y = \tfrac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad (3.4b)$$

and rotation about axis x

$$\omega_x = \tfrac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad (3.4c)$$

If the fluid is irrotational then it should have

$$\left.\begin{array}{llll} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 & , & \text{i.e.} & \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0 & , & \text{i.e.} & \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0 & , & \text{i.e.} & \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \end{array}\right\} \qquad (3.5)$$

**EXAMPLE 3.1**

In a flow field the velocity components u, v and w (m/s) in terms of distances in the directions x,y and z (m) respectively are given as follows

$u = 3xy + z$
$v = 2x^2 + y^2$
$w = 5yz$

Determine

a) The rotational speeds of the fluid elements,
b) the components of the acceleration in the directions x, y and z,and

c) the magnitude of both the velocity and the acceleration at $x = 3, y = -2$ and $z = 4$.

---

Data of the Problem

$$* \; u = 3xy + z, \quad v = 2x^2 + y^2, \quad w = 5yz \qquad \text{(I)}$$

Requirements

* $\omega_x$, $\omega_y$ and $\omega_z$
* $a_x$, $a_y$ and $a_z$
* c and a at (x, y, z) = (3, -2, 4)

Solution

The rotational speeds $\omega_x$, $\omega_y$ and $\omega_z$ are given by Eq. (3.4) as follows:

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \frac{1}{2}(5z - 0) = \frac{5}{2} z$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{1}{2}(1 - 0) = \frac{1}{2}$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(4x - 3x) = \frac{1}{2}x$$

The acceleration components $a_x$, $a_y$ and $a_z$ are given by Eqs. (3.3). When Eqs. (I) are used to replace the partial derivatives and the velocities u, v and w respectively of Eqs. (3.3), the expressions for $a_x$, $a_y$ and $a_z$ become

$$\begin{aligned}
a_x &= 0 + (3xy + z)(3y) + (2x^2 + y^2)(3x) + (5yz)(1) \\
&= 9xy^2 + 3yz + 6x^3 + 3xy^2 + 5yz \\
&= 12xy^2 + 8yz + 6x^3 \\
a_y &= 0 + (3xy + z)(4x) + (2x^2 + y^2)(2y) + (5yz)(0) \\
&= 12x^2y + 4xz + 4x^2y + 2y^3 \\
&= 4xz + 16x^2y + 2y^3 \\
a_z &= 0 + (3xy + z)(0) + (2x^2 + y^2)(5z) + (5yz)(5y) \\
&= 10x^2z + 5y^2z + 25y^2z \\
&= 10x^2z + 30y^2z
\end{aligned}$$

The velocity components u, v and w at (x, y, z) = (3, −2, 4) have the following values

$$\begin{aligned}
u &= 3 \times 3 \times (-2) + 4 &&= -14 \text{ m/s} \\
v &= 2 \times (3)^2 + (-2)^2 &&= 22 \text{ m/s} \\
w &= 5 \times (-2) \times 4 &&= -40 \text{ m/s}
\end{aligned}$$

Therefore the magnitude of the velocity becomes

$$C = \sqrt{u^2 + v^2 + w^2} = 47.75 \quad m/s$$

Similarly, the values of $a_x$, $a_y$ and $a_z$ at (3, −2, 4) are

$$a_x = 12 \times 3 \times (-2)^2 + 8 \times (-2) \times 4 + 6 \times (3)^3 = 242\ m/s^2$$
$$a_y = 4 \times 3 \times 4 + 16 \times (3)^2 \times (-2) + 2 \times (-2)^3 = -256\ m/s^2$$
$$a_z = 10 \times (3)^2 \times 4 + 30 \times (-2)^2 \times 4 = 840\ m/s^2$$

and the magnitude of the acceleration at (3, −2, −4) becomes

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2} = 910.88 \quad m/s^2$$

## 3 – 4 – Flow Rates Through Areas

If the velocity vector "C" in a uniform velocity field, see Fig. (3.2a), is perpendicular to the area "A" through which the fluid is being discharged, then the volume and mass flow rates through this area are given respectively as follows

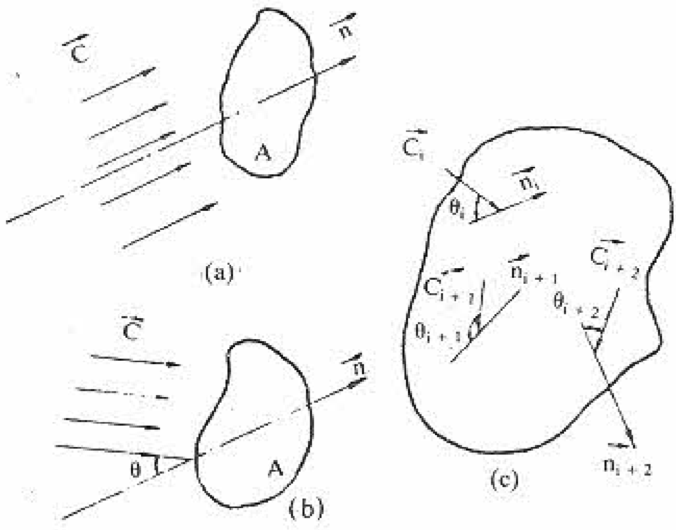



*Fig. 3.2 Flow rates through areas*

$$\dot{V} = \frac{dV}{dt} = C\ A \tag{3.6a}$$

and

$$\dot{m} = \frac{dm}{dt} = C\ A \tag{3.6b}$$

where t represents time. If the velocity vector is inclined at an angle $\theta$ to the axis of the area, as shown in Fig. (3.2b), then the flow rates are given as follows

$$\dot{V} = C\ A\ \cos\ \theta \tag{3.7a}$$

$$\dot{m} = \rho C\ A\ \cos\ \theta \tag{3.7b}$$

Also, when the magnitude and the inclination of the velocity vector varies from one part of the area to the other as shown in Fig. (3.2c), then

$$\dot{V} = \int_A C\ \cos\ \theta\ dA \tag{3.8a}$$

$$\dot{m} = \int_A \rho C\ \cos\ \theta\ dA \tag{3.8b}$$

The component of the velocity "C cos $\theta$" which is normal to the infinitesimal area "dA" may be denoted as "$C_n$". Thus Eqs. (3,8a) and (3.8b) may be written in the form

$$\dot{V} = \int_A C_n\ dA \tag{3.9a}$$

$$\dot{m} = \int_A \rho C_n\ dA \tag{3.9b}$$

**EXAMPLE 3.2**

The velocity distribution of water flowing in a pipe is given as follows

$$\frac{C}{5} = 1 - \left(\frac{r}{R}\right)^2$$

where R = 0.1m is the radius of the pipe, r is the radial distance measured from the centerline in meters, and C m/s is the axial velocity at any distance r. Determine the mass flow rate of the water.

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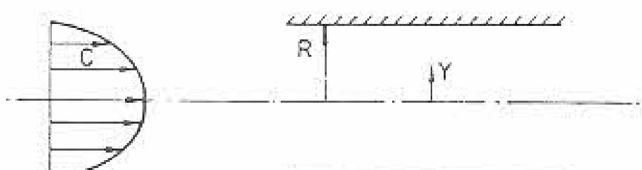
Problem Description




Data of the Problem

* $\frac{C}{5} = 1 - \left(\frac{r}{R}\right)^2$

r and R in m, C in m/s

* $R = 0.1$ m
* Fluid is water (i.e. $\rho = 1000\ kg/m^3$)

Requirement

* mass flow rate, $\dot{m}$

Solution

From Eq. (3.9b), we have

$$\dot{m} = \int_A \rho C_n \ dA$$

$$\dot{m} = \int 5\rho(1 - \frac{r^2}{R^2})\ (2\pi\ rdr)$$

$$= 10\pi\rho \int_o^R (r - \frac{r^3}{R^2})\ dr$$

$$= 10\ \pi\rho\ (\frac{r^2}{2} - \frac{r^4}{4R^2})\Big|_o^R$$

$$= \frac{5}{2}\ \pi\rho\ R^2$$

Substituting 0.1m for R and 1000 $kg/m^3$ for $\rho$ we get

$$\dot{m} = \frac{5}{2} \times \pi \times 1000 \times (0.1)^2$$

$$= 78.54 \quad kg/s$$

---

**EXAMPLE 3.3**

At the exit of an open rectangular channel bend the velocity at water surface varies with the width from 4.0m/s at the outer wall, to 1.5m/s at the inner wall according to the relation

$$C_s = 4 - \alpha \sqrt{x}$$

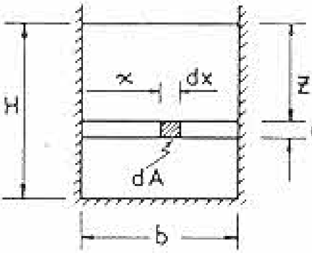
where "$C_s$" is the velocity at any point on water surface, "$\alpha$" is a constant and "x" is the distance of that point from the outer wall. The velocity "C" at any depth in this cross section varies according to the relation,

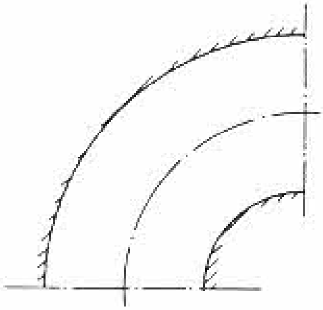
$$C = C_s - z^{0.8}$$

where "$C_s$" is the velocity at water surface vertically above the point under consideration, and "z" is the depth of that point below water surface. If the width of the channel is 1.0m and the depth of water in the channel is 1.5m, calculate the volume flow rate. Note that velocities are perpendicular to the exit area of the channel and the boundary layer effects are neglected.

---

Problem Description



Data of the Problem

* $H = 1.5\text{m}$ , $b = 1.0\text{m}$
* velocity distribution at surface: $C_s = 4 - \alpha \sqrt{x}$ (I)
* velocity distribution at any depth: $C = C_s - z^{0.8}$ (II)

- at $x = 0$ m $\quad C_s = 4$ m/s
  at $x = 1$ m $\quad C_s = 1.5$ m/s (III)

Requirement

* the volume flow rate in the channel

Solution

From Eq. (3.9a), the volume flow rate is given as follows

$$\dot{V} = \int_A C \, dA = \int_0^H \int_0^b (C_s - z^{0.8}) \, dx dz$$

$$= \int_0^{1.5} \int_0^{1} (4 - \alpha \sqrt{x} - z^{0.8}) \, dx dz$$

$$= \int_0^{1.5} \left(4x - \frac{2}{3} \alpha x \sqrt{x} - z^{0.8}\right) \Big|_0^1 dz$$

$$= \int_{0}^{1.5} (4 - \frac{2}{3}\alpha - z^{0.8})\ dz$$

$$= (4z - \frac{2}{3}\alpha z - \frac{1}{1.8} z^{1.8}) \Big|_{0}^{1.5}$$

$$= (6 - \alpha) - 1.15$$

$$= 4.85 - \alpha \tag{IV}$$

Now the constant a is to be calculated from Eq. (I) and its boundary condition Eq. (III). The second boundary condition yields

$$1.5 = 4 - \alpha \sqrt{1}$$

or,

$$\alpha = 2.5\ m^{\frac{1}{2}}/s \tag{V}$$

Therefore, Eq. (IV) becomes

$$\dot{V} = 2.35\ m^3/s$$

---

## 3 – 5 – Conservation of Mass

Conservation of mass is a law of nature which implies that "In the absence of nuclear reactions, mass, can neither be created nor be destroyed and the sum of all masses in an isolated system must remain the same at all times."

### 3 – 5 – 1 – Application of conservation of mass to a finite control volume

A control volume is an arbitrary volume of an arbitrary shape in space through which fluid may flow. The control volume may be fixed or moving in the space. The boundaries of the control volume may be rigid or flexible. When a control volume has rigid boundaries, its volume becomes constant, whereas a control volume with flexible boundaries has a variable volume. Examples of control volumes with rigid boundaries are a tank of water with input and/or output flow, a running automobile with air input and exhaust output, etc.. Among the examples of control volumes with flexible boundaries are the human lungs, automobile tires and the childrens balloons.

Consider the finite volume shown in Fig. (3.3) into which fluid flows through different inlets and out of which fluid also flows through different outlets and in different directions.

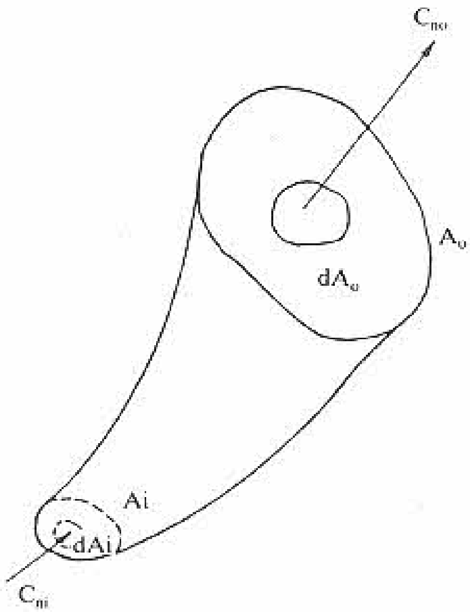



*Fig. 3.3 Application of conservation of mass to a finite control volume*

The law of conservation of mass would then require that the rate of increase of the mass of the control volume with respect to time should be equal to the difference between the rate at which the fluid flows into the control volume and the rate at which the fluid flows out of the control volume, i.e.

$$\dot{m}_{cv} = \dot{m}_i - \dot{m}_o \tag{3.10}$$

Where the suffixes i,o and cv refer to inlet, outlet and control volume respectively. Now the rate at which the fluid flows into the control volume is

$$\dot{m}_i = \int \rho_i \, C_{ni} \, dA_i \tag{3.11a}$$

and the rate at which fluid leaves the control volume is

$$\dot{m}_o = \int \rho_o \, C_{no} \, dA_o \tag{3.11b}$$

The rate, at which the mass of the system increases, is given as follows

$$\dot{m}_{cv} = \frac{d}{dt} ( \rho_{cv} V_{cv} ) \tag{3.11c}$$

where $\rho_{cv}$ is the mean density of the control volume. The value of $\rho_{cv}$ may be different from the densities of the fluid flowing into or out of the control volume. Substitution by Eqs. (3.11a), (3.11b) and (3.11c) into Eq. (3.10) yields

$$\int \rho_i \; C_{ni} \; dA_i - \int \rho_o \, C_{no} \; dA_o = \frac{d}{dt} \left( \rho_{cv} \, V_{cv} \right) \tag{3.12}$$

In some cases, the above equation can be simplified. Some of these cases are summarized below.

In a control volume of constant volume Eq. (3.12) takes the form

$$\int \rho_i \; C_{ni} \; dA_i - \int \rho_o \; C_{no} \; dA_o = V_{cv} \frac{d\rho_{cv}}{dt} \tag{3.13}$$

But, if the density of control volume is constant and its volume is changing with time, the conservation of mass law becomes

$$\int \rho_i \; C_{ni} \; dA_i - \int \rho_o \, C_{no} \; dA_o = \rho_{cv} \frac{dV_{cv}}{dt} \tag{3.14}$$

Another special case may be of interest, this is the case where the control volume is constant and the density at both inlet and outlet is constant, then we get.

$$\rho_i \int C_{ni} \; dA_i - \rho_o \int C_{no} \; dA_o = V_{cv} \frac{d\rho_{cv}}{dt} \tag{3.15}$$

Also, a constant control volume with constant density and uniform velocity and density at both inlet and outlet is simplified to the following

$$\rho_i \; C_{ni} \; A_i = \rho_o \; C_{no} \; A_o \tag{3.16}$$

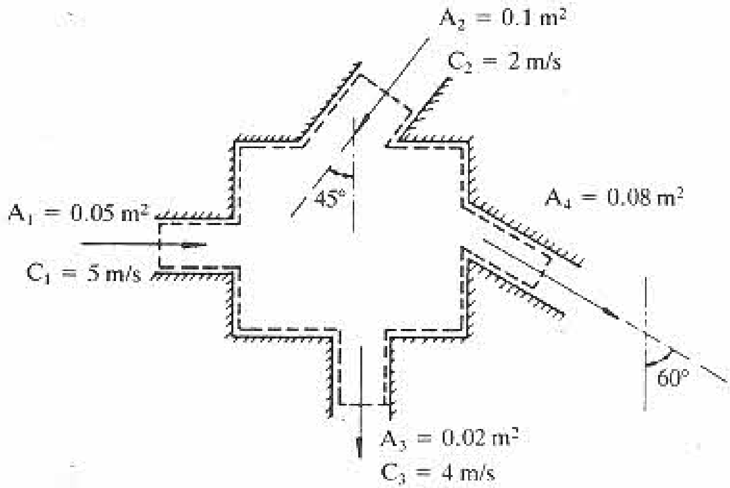
and if $\rho_i = \rho_o$ in the above expression, it can be further simplified to

$$C_{ni} \; A_i = C_{no} \; A_o \tag{3.17}$$

**EXAMPLE 3 – 4**

Water flows steadily through the configuration shown in the problem description. Determine the velocity $C_4$.

Problem Description




Data of the Problem

* flow of water ($\rho = 1000$ kg/m³)
* Areas and velocities as shown in the Problem Description

Requirement

* exit velocity $C_4$

Solution

Since the flow is incompressible and steady, the application of the conservation of mass on the control volume bounded by the broken line gives

$$\dot{V}_1 + \dot{V}_2 = \dot{V}_3 + \dot{V}_4$$

where $\dot{V}$ is the volume flow rate. This is rewritten as follows

$$\dot{V}_4 = \dot{V}_1 + \dot{V}_2 - \dot{V}_3$$

$$= C_1 A_1 + C_2 A_2 - C_3 A_3$$

$$= 5 \times 0.05 + 2 \times 0.1 - 4 \times 0.02$$

$$= 0.037 \quad m^3/s$$

$$C_4 = \frac{\dot{V}_4}{A_4} = \frac{0.037}{0.08} = 4.625 \quad m/s$$

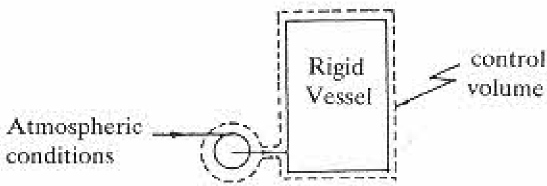
**EXAMPLE 3 – 5**

A rigid vessel of $2m^3$ volume is being filled with air by a compressor which draws air from the atmosphere at a constant rate of $0.2\ m^3/min$ and discharges it into the vessel. The temperature of the air in the vessel may be assumed to remain constant during the compression process, due to heat transfer to the surroundings. Find the time taken to increase the pressure in the vessel from the atmospheric pressure to $500\ kN/m^2$.

Consider air to be a perfect gas that follows the law $PV = mRT$. Take atmospheric pressure as $100\ kN/m^2$, atmospheric temperature as 15°C and R for atmospheric air as 287 J/kgK.

---

Problem Description




Data of the Problem

* Atmospheric conditions

  $P_a = 10^5\ N/m^2$

  $T_a = 288\ K$

  $R = 287\ J/kg\ K$
* input volume flow rate $= 0.2 \times (1/60) = 1/300\ m^3/s$
* volume of the vessel, $V_v = 2\ m^3$
* vessel temperature is kept constant and is equal to atmospheric temperature,

  $T_v = T_a$ (I)
* air to be as a perfect gas, $P = \rho RT$ (II)

Requirement

* the time required to increase the pressure in the vessel from atmospheric pressure to $5 \times 10^5\ N/m^2$

Solution

From Eq. (3.10), the conservation of mass yields

$$\dot{m}_{cv} = \dot{m}_i - \dot{m}_o$$

but since there is no mass flow out of the vessel, then

$$\dot{m}_{cv} = \dot{m}_i$$

or,
$$\frac{d}{dt}\left( \rho_v V_v \right) = \rho_a \dot{V}_a \qquad \text{(III)}$$

where the subscripts v and a refer to the vessel and atmospheric air respectively. Assuming air to be a perfect gas, Eq. (III) becomes

$$\frac{d}{dt}\left( \frac{P_v}{RT_v} V_v \right) = \rho_a \dot{V}_a \qquad \text{(IV)}$$

For a rigid vessel, i.e. $V_c$ = const, with constant temperature Eq. (IV) takes the following form

$$\frac{dP_v}{dt} = P_a \frac{\dot{V}_a}{V_v}$$

or,
$$\int_{P_a}^{P_v} dP_v = P_a \frac{\dot{V}_a}{V_v} \int_0^t dt$$

or,
$$t = \frac{V_v}{\dot{V}_a} \frac{P_v - P_a}{P_a}$$

$$= \frac{2 \times 300}{1} \times \frac{5\ 10^5 - 10^5}{10^5}$$

$$= 2400 \text{ seconds}$$
$$= 20 \text{ minutes}$$

---

**3 – 5 – 2 – Application of conservation of mass to an infinitesimal control volume; continuity equation**

Consider a control volume, Fig. (3.4), in a fluid continuum that is made up of a parallelepiped fluid element of fixed dimensions dx, dy and dz and fixed in space. Since mass can neither be created nor be destroyed, hence in any period of time the rate of change of the mass of this control volume is equal to the difference between the sum of the masses that entered and those that left the control volume.

Let us now consider u, v, w and ρ to be the velocity components in the directions x, y, z and density respectively, at the geometrical centre of the fluid element. The rate of mass flowing into the control volume in the direction x is equal to

$$\dot{m}_{xi} = \rho u \, dy \, dz - \frac{\partial(\rho u)}{\partial x} \, dy \, dz \, \frac{dx}{2} \tag{3.18a}$$

The rate of mass flowing out of the control volume in direction x is equal to

$$\dot{m}_{xo} = \rho u \, dy \, dz + \frac{\partial(\rho u)}{\partial x} \, dy \, dz \, \frac{dx}{2} \tag{3.18b}$$

The net change of the mass of the control volume due to flow in direction x is the difference of the above two quantities, i.e.

$$\dot{m}_{xi} - \dot{m}_{xo} = - \frac{\partial}{\partial x} (\rho u) \, dx \, dy \, dz \tag{3.19a}$$

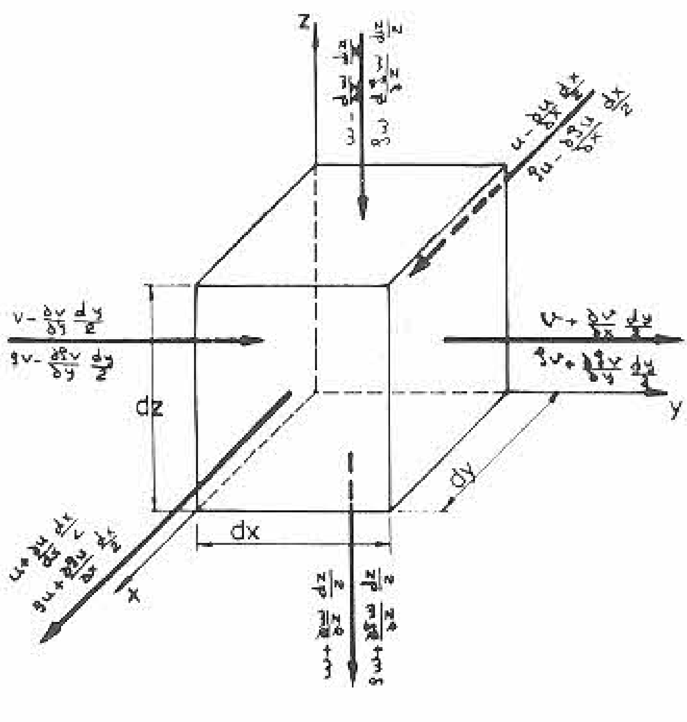



*Fig. 3.4 Application of conservation of mass to an infinitesimal control volume*

Following the same analysis the change of the mass of the control volume due to flow in directions y and z can respectively be given as follows

$$\dot{m}_{yi} - \dot{m}_{yo} = - \frac{\partial}{\partial y} (\rho v) \, dx \, dy \, dz \tag{3.19b}$$

$$\dot{m}_{zi} - \dot{m}_{zo} = - \frac{\partial}{\partial z} (\rho w) \, dx \, dy \, dz \tag{3.19c}$$

Also, the rate of change of the mass in the control volume is given as follows

$$\dot{m}_{cv} = \frac{\partial \dot{m}}{\partial t}_{cv} = \frac{\partial}{\partial t} (\rho) \, dx \, dy \, dz \tag{3.20}$$

The conservation law, Eq. (3.10), can be rewritten in terms of Eqs. (3.19) and (3.20) as follows

$$\begin{aligned} \frac{\partial}{\partial t} (\rho) \, dx \, dy \, dz = & - \frac{\partial}{\partial x} (\rho u) \, dx \, dy \, dz \\ & - \frac{\partial}{\partial y} (\rho v) \, dx \, dy \, dz \\ & - \frac{\partial}{\partial z} (\rho w) \, dx \, dy \, dz \end{aligned}$$

or,

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0 \tag{3.21}$$

which is the continuity equation for compressible three dimensional flow. Some cases can be derived to simplify Eq. (3.21). For example, at steady state conditions Eq. (3.21) becomes

$$\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0 \tag{3.22}$$

A further simplification can be obtained if the flow is two dimensional, thus

$$\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0 \tag{3.23}$$

Furthermore for incompressible flow, i.e. constant density, it becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.24}$$

**EXAMPLE 3.6**

The following velocity components are claimed to be possible flow cases of water continuum. Check this claim neglecting any possible variations in the density of water

a) $u = xy$, $v = yz$, $w = yz + z^2$

b) $u = x + y + z$, $v = x - y + z$, $w = x + y$

---

Data of the Problem

* Claiming two possible flow cases of water continuum

  a) $u = xy$, $v = yz$, $w = yz + z^2$

  b) $u = x + y + z$, $v = x - y + z$, $w = x + y$

Requirement

* Check these two claims

Solution

If the claim is correct, the velocity distribution should satisfy the continuity equation, Eq. (3.21). Since water can be considered as incompressible fluid, Eq. (3.21) can be simplified to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{(I)}$$

a) Substitute by the velocity distribution into Eq. (I), this yields

$$(y) + (z) + (y + 2z) = 2y + 3z$$
$$\neq 0$$

i.e. the given velocity distribution can not be a possible flow case.

b) Similarly, the continuity equation, Eq. (I), becomes

$(1) - (1) + 0 = 0$

i.e. the velocity distribution is a possible flow case.

---

**EXAMPLE 3.7**

In a steady incompressible forced vortex flow the velocity components in directions x,y and z are u,v and w respectively. If w is independent of x and y, and u and v are given as follows

$u = -\Omega y$

and

$v = \Omega x$

where $\Omega$ is the angular velocity. Find the expression for w. Also, calculate the rotational speeds around the coordinates x, y and z.

---

Data of the Problem

* steady, incompressible flow
* $w \neq w(x, y)$ (I)
* $u = -\Omega y,\ v = \Omega x$ (II)

Requirements

* the expression for velocity component in direction z, w
* rotational speeds; $\omega_x$ , $\omega_y$ , $\omega_z$

Solution

The continuity equation for steady incompressible flow, Eq. (3.21), is simplified to the following

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{(III)}$$

Substituting by Eq. (II) in the above equation, we get

$$0 + 0 + \frac{\partial w}{\partial z} = 0$$

or

$w = f(x, y)$

but recalling Eq. (I), then (IV)

$w = \text{constant}$

The rotational speeds of the fluid elements are given by Eq. (3.4). These give the following

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = 0$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = 0$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \Omega$$

which means that the flow is rotational only around the axis z, and its rotational speed is equal in magnitude and direction to the angular speed $\Omega$.

## 3 – 6 – Momentum Principle

In a general control volume moving with velocity $C_{cv}$ with mass efflux and mass influx, the momentum principle states that the rate of change in momentum of the control volume in a certain direction is equal to the rate of momentum influx in the same direction, minus the rate of momentum efflux in the same direction, plus the sum of forces acting on the control volume in the same direction. For example, the momentum principle in the direction x becomes

Rate of change in momentum of C.V. in direction x
= Rate of momentum influx in direction x
− Rate of momentum efflux in direction x
+ Sum of forces acting on C.V. in direction x (3.25a)

or
$$\dot{G}_{x,cv} = \dot{G}_{x,i} - \dot{G}_{x,o} + \Sigma F_x \tag{3.25b}$$

where $\dot{G}$ represents the rate of momentum and $\Sigma$ F is the sum of forces acting on the control volume. The subscripts x, cv, i and o refer to direction x, control volume, input and output respectively. The term $\Delta\dot{G}$ means the rate of change of momentum. In the following two subsections, the momentum principle, Eq. (3.25), is to be applied to both a finite and an infinitesimal control volume.

### 3 − 6 − 1 − Application of momentum principle to a finite control volume

Consider the control volume in Fig. (3.5), the individual terms of Eq. (3.25b) can be written as follows

$$\dot{G}_{x,cv} = \frac{d}{dt}(m_{cv}\, u_{cv}) \tag{3.26a}$$

$$\dot{G}_{x,i} = \int_{Ai} u_i\, d\dot{m}_i = \int_{Ai} \rho_i u_i\, C_{ni}\, dA_i \tag{3.26b}$$

$$\dot{G}_{x,o} = \int_{Ao} u_o\, d\dot{m}_o = \int_{Ao} \rho_o u_o\, C_{no}\, dA_o \tag{3.26c}$$

Therefore Eq. (3.25b) becomes

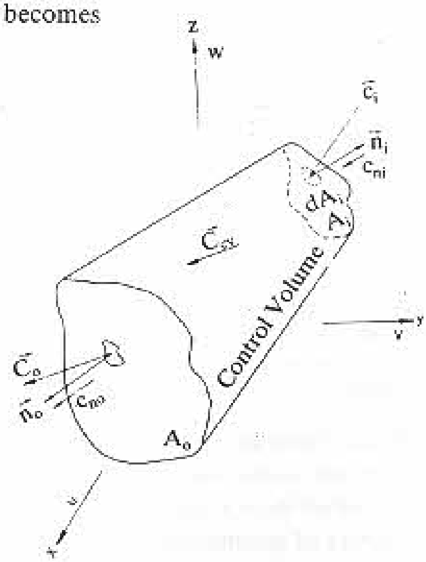



*Fig. 3.5 Momentum principle as applied to a finite control volume.*

$$\frac{d(m_{cv}u_{cv})}{dt} = \int_{Ai} \rho_i u_i C_{ni} dA_i - \int_{A_o} \rho_o u_o C_{no} dA_o + \Sigma F_x$$

or

$$\Sigma F_x = m_{cv} a_{x,cv} + u_{cv} \dot{m}_{cv} + \int_{A_o} \rho_o u_o C_{no} dA_o - \int_{A_i} \rho_i u_i C_{ni} dA_i \tag{3.27a}$$

where $a_{x,cv} = du_{cv}/dt$ is the acceleration of the control volume in the direction x, $C_{ni}$ and $C_{no}$ are the relative inlet and exit normal velocity component respectively. Similarly, the application of the momentum principle in the directions y and z respectively gives

$$\Sigma F_y = m_{cv} a_{y,cv} + v_{cv} \dot{m}_{cv} + \int_{A_o} \rho_o v_o C_{no} dA_o - \int_{A_i} \rho_i v_i C_{ni} dA_i \tag{3.27b}$$

and

$$\Sigma F_z = m_{cv} a_{z,cv} + w_{cv} \dot{m}_{cv} + \int_{A_o} \rho_o w_o C_{no} dA_o - \int_{A_i} \rho_i w_i C_{ni} dA_i \tag{3.27c}$$

If the flow has a uniform velocity at both inlet and exit of the control volume, Eqs (3.27) are simplified to

$$\Sigma F_x = m_{cv} a_{x,cv} + u_{cv} \dot{m}_{cv} + \dot{m}_o u_o - \dot{m}_i u_i \tag{3.28a}$$

$$\Sigma F_y = m_{cv} a_{y,cv} + v_{cv} \dot{m}_{cv} + \dot{m}_o v_o - \dot{m}_i v_i \tag{3.28b}$$

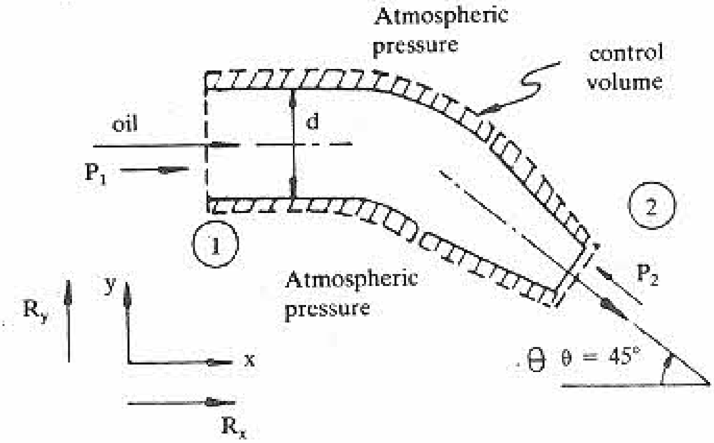
$$\Sigma F_z = m_{cv} a_{z,cv} + w_{cv} \dot{m}_{cv} + \dot{m}_o w_o - \dot{m}_i w_i \tag{3.28c}$$

Note that the rate of change of the mass of the control volume $\dot{m}_{cv}$, may be given by Eq. (3.10).

**EXAMPLE 3.8**

A rate of 200kg/s of oil (sp. gr. = 0.8) flows through a 45° elbow of diameter 10 cm. The pressure of oil at the inlet section is 300 kN/$m^2$ abs. A nozzle of area ratio 2:1 is connected to the elbow exit. The nozzle discharges directly to the atmosphere at 100 kN/$m^2$ abs. Calculate the force required to hold the assembly of elbow and nozzle at rest.

Problem Description




Data of the Problem

* flow of oil, sp. gr. = 0.8, through the assembly shown in Problem Description
* $P_1 = 3 \times 10^5$ N/m², $P_2 = 10^5$ N/m²
* $\dot{m}_{oil} = 200$ kg/s
* d = 0.1 m
* $A_1/A_2 = 2$

Requirement

* the force to hold the assembly of elbow and nozzle at rest

Solution

Applying the momentum principle in the direction x, Eq. (3.28a), to the shown control volume yields

$$\Sigma F_x = m_{cv}\, a_{x,cv} + u_{cv}\, \dot{m}_{cv} + \dot{m}_o\, u_o - \dot{m}_i\, u_i \tag{I}$$

but

$$a_{x,cv} = u_{cv} = 0 \qquad \text{(assembly at rest)}$$

$$\Sigma F_x = (P_1 - P_{atm})\, A_1 + R_x$$

$$u_o = C_2 \cos\theta$$

$$u_i = C_1$$

Then Eq. (I) becomes

$$(P_1 - P_{atm})\, A_1 + R_x = \dot{m}_o\, C_2 \cos\theta - \dot{m}_i\, C_1 \qquad \text{(II)}$$

At steady, incompressible flow the conservation of mass leads to

$$\dot{m}_o = \dot{m}_i = \dot{m}$$

i.e.

$$C_1 A_1 = C_2 A_2$$

or

$$C_2 = \frac{A_1}{A_2} C_1$$

Therefore, Eq. (II) becomes

$$(P_1 - P_{atm})\, A_1 + R_x = \dot{m}\, C_1 \left( \frac{A_1}{A_2} \cos\theta - 1 \right)$$

$$R_x = \dot{m}\, C_1 \left( \frac{A_1}{A_2} \cos\theta - 1 \right) - (P_1 - P_{atm}) A_1 \qquad \text{(III)}$$

But, Since

$$\dot{m} = \rho\, C_1\, A_1$$

Then

$$C_1 = \frac{\dot{m}}{\rho A_1} = \frac{200}{800 \times \frac{\pi}{4} (0.1)^2} = 31.83 \text{ m/s}$$

and Eq. (III) becomes

$$R_x = 200 \times 31.83\, (2 \cos 45 - 1) - (3 - 1) \times 10^5 \times \frac{\pi}{4} (0.1)^2$$
$$= 1066.087 \text{ N}$$

Applying the momentum principle in the y-direction yields

$$\Sigma F_y = m_{cv}\, a_{y,cv} + v_{cv}\, \dot{m}_{cv} + \dot{m}_o\, v_o - \dot{m}_i\, v_i$$

which can be simplified to

$$R_y = - \dot{m}\, C_2 \sin\theta = - m\, C_1 \frac{A_1}{A_2} \sin\theta$$
$$= - 200 \times 31.83 \times 2 \times \sin 45$$
$$= - 9002.88 \text{ N}$$

and, therefore

$$R = \sqrt{R_x^2 + R_y^2}$$
$$= \sqrt{(1066.08)^2 + (9002.88)^2} = 9065.78 \quad N$$

and at an angle $\phi$ from the horizontal given as follows

$$\phi = \tan^{-1} \frac{-9002.88}{1066.08} = -83.24^{\circ}$$

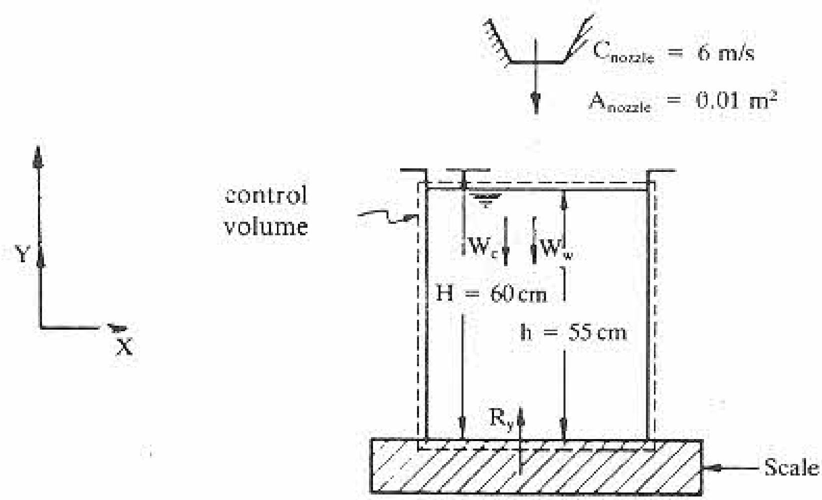
---

**EXAMPLE 3.9**

A rigid container 60 cm high with an inside cross-sectional area of 0.1 $m^2$ weighs 2 kg when empty. The container is placed on a scale and water flows in through an opening *in the top as shown* in the problem description. The pressure is atmospheric at the inlet and around the tank. Determine the reading of the scale when the water height in the tank is 55 cm (neglect change in the velocity of flow between nozzle exit and water surface in the tank).

---

Problem Description




Data of the Problem

* water tank on a scale as shown in the Problem Description
* dimension of the container, $A_c$ = 0.1 $m^2$, H = 0.6m
* weight of the empty container, $W_c$ = 2 kg
* nozzle condition: $A_N$ = 0.01 $m^2$ $C_N$ = 6 m/s,

Requirement

* The reading of the scale when the water height is 55 cm, i.e. h = 55 cm

Solution

Applying the momentum principle in direction y, Eq. (3.28 – b). gives

$$\Sigma F_y = m_{cv} \cancelto{0}{a_{y,cv}} + \cancelto{0}{v_{cv}} \dot{m}_{cv} + \dot{m}_o \cancelto{0}{v_{cv}} - \dot{m}_i v_i$$

or,

$$\Sigma F_y = - \dot{m}_i v_i$$

or,

$$R_y - W_y - W_w = - \rho_w A_N C_N (- C_N)$$

where $W_w$ is the weight of the water in the tank. Therefore

$$R_y - 2g - \rho_w Ahg = \rho_w A_N C_N^2$$

or,

$$R_y = 10^3 \times 0.01 \times (6)^2 + 2\times9.8 + 10^3\times0.1 \times 0.55 \times 9.8$$
$$= 918.6 \text{ N}$$

the scale reading = $R_y / g = 918.6 / 9.8$
$= 93.7$ kg

Note that if there is no momentum input in the direction y the reading of the scale would be

scale reading (at no flow) $= (W_c + W_w) / g$
$= 2 + 10^3 \times 0.1 \times 0.55$
$= 57$ kg

---

### 3 – 6 – 2 – Application of momentum principle to an infinitesimal control volume; Euler's equations

Here the momentum principle is to be applied to an infinitesimal control volume. Select a control volume, fixed in space, with fixed infinitesimal dimensions dx, dy and dz. Velocity components at the centre of geometry of the control volume are assumed to be u, v and w in directions x,y and z respectively. For easy reference, the different faces of the control volume are numbered as shown in Fig. (3.6).

The rate of momentum influx to the control volume in direction x is the sum of the rates of momentum influx through faces "1", "3" and "5" of the control volume; denoted as $\dot{G}_{x,1}$, $\dot{G}_{x,3}$ and $\dot{G}_{x,5}$ respectively. The latter are given in the forms

$$
\left.
\begin{aligned}
\dot{G}_{x,1} &= \{ \rho u^2 - \frac{\partial}{\partial x} (\rho u^2) \frac{dx}{2} \} \, dy \, dz \\
\dot{G}_{x,3} &= \{ \rho uv - \frac{\partial}{\partial y} (\rho uv) \frac{dy}{2} \} \, dx \, dz \\
\dot{G}_{x,5} &= \{ \rho uw - \frac{\partial}{\partial z} (\rho uw) \frac{dz}{2} \} \, dx \, dy
\end{aligned}
\right\} \qquad (3.29a)
$$

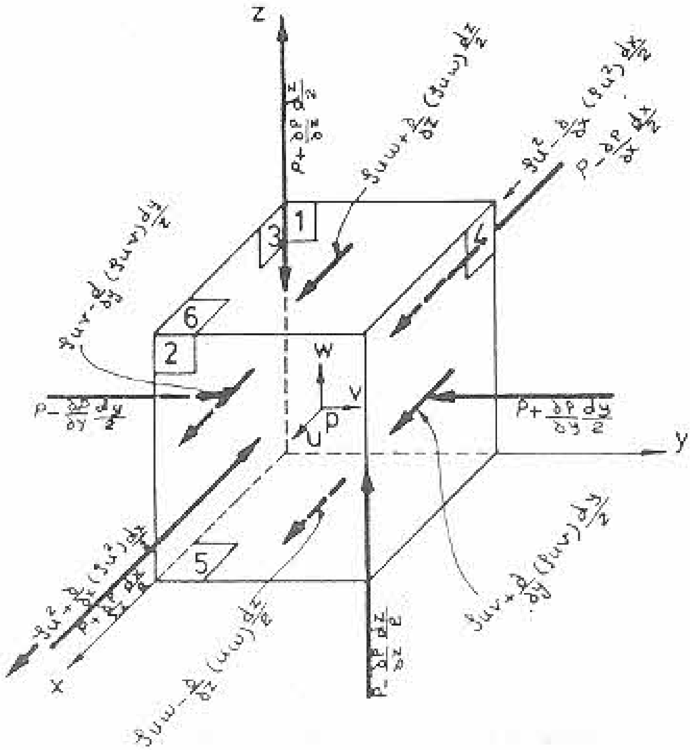



*Fig. 3.6 Rates of momentum influx and efflux*

Also, the rate of momentum efflux from the control volume, in direction, x, is the sum of the rates of momentum efflux from faces "2", "4" and "6" of the control volume; $G_{x,2}$, $G_{x,4}$ and $G_{x,6}$ respectively. These take the following forms

$$
\left.
\begin{aligned}
\dot{G}_{x,2} &= (\rho\, u^2 + \frac{\partial}{\partial x}(\rho u^2)\frac{dx}{2})\ dy\ dz \\
\dot{G}_{x,4} &= (\rho\, uv + \frac{\partial}{\partial y}(\rho uv)\frac{dy}{2})\ dx\ dz \\
\dot{G}_{x,6} &= (\rho\, uw + \frac{\partial}{\partial z}(\rho\, uw)\frac{dz}{2})\ dx\ dy
\end{aligned}
\right\} \tag{3.29b}
$$

The net rate of momentum influx and efflux in the direction $x$, $\dot{G}_{x,i} - \dot{G}_{x,0}$, can be obtained by subtracting the sum of Eqs. (3.29b) from the sum of Eqs. (3.29a) which yields

$$
\dot{G}_{x,i} - \dot{G}_{x,o} = -\left\{ \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw) \right\} dx\ dy\ dz \tag{3.30}
$$

The above equation is the value of the first and second terms in the right hand side of Eq. (3.25b). The left hand side of Eq. (3.25b), i.e. the rate of change of momentum of the control volume in direction x is calculated as follows

$$
\begin{aligned}
\dot{G}_{x,cv} &= \frac{\partial}{\partial t}(\rho\ dx\ dy\ dz\ u), \\
&= \frac{\partial}{\partial t}(\rho u)\ dx\ dy\ dz
\end{aligned} \tag{3.31}
$$

Only the last term in the right hand side of Eq. (3.25b) is still unknown. The resultant force in direction x is the sum of two parts. The first part is the net pressure force in direction x which is given as follows

$$
\begin{aligned}
(P - \frac{\partial p}{\partial x}\frac{dx}{2})\ dy\ dz - (P + \frac{\partial p}{\partial x}\frac{dx}{2})\ dy\ dz \\
= -\frac{\partial p}{\partial x}\ dx\ dy\ dz
\end{aligned}
$$

The second part of the resultant force is the net external force acting on the control volume in direction x which can be assumed as $\beta_x$ per unit mass of the fluid. Therefore the resultant force in direction x becomes

$$
\Sigma F_x = \rho \beta_x\ dx\ dy\ dz - \frac{\partial p}{\partial x}\ dx\ dy\ dz \tag{3.32}
$$

Equation (3.32) completes the calculations of the various terms of the momentum principle given by Eq. (3.25b). Substitution by Eqs. (3.30) to (3.32) into Eq. (3.25b) gives

$$
\begin{aligned}
\rho \beta_x - \frac{\partial p}{\partial x} = \frac{\partial}{\partial t}(\rho u) + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} \\
+ \frac{\partial(\rho uw)}{\partial z}
\end{aligned} \tag{3.33a}
$$

where the volume of the control volume dx dy dz is eliminated from the equation. The equation can also be written in the form

$$\rho\, \beta_x - \frac{\partial p}{\partial x} = u \left[ \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] + \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \qquad (3.33b)$$

The sum of the terms between the first square brackets is equal to zero from the continuity equation, Eq. (3.21). Therefore, Eq. (3.33b) becomes

$$\beta_x - \frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \qquad (3.34a)$$

By analogy to the above procedure, it can be shown that the momentum equation in directions y and z can be written as follows

$$\beta_y - \frac{1}{\rho} \frac{\partial p}{\partial y} = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \qquad (3.34b)$$

and

$$\beta_z - \frac{1}{\rho} \frac{\partial p}{\partial z} = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \qquad (3.34c)$$

where $\beta_y$ and $\beta_z$ are the net external forces per unit mass in directions y and z respectively.

Equations (3.34) are known as Euler's equations in directions x,y and z respectively. The right hand sides of these equations are the accelerations of the fluid in directions x,y and z as given by Eq. (3.3). Euler's equations may also be stated in the form

$$\left.\begin{aligned} \beta_x - \frac{1}{\rho} \frac{\partial p}{\partial x} &= a_x \\ \beta_y - \frac{1}{\rho} \frac{\partial p}{\partial y} &= a_y \\ \beta_z - \frac{1}{\rho} \frac{\partial p}{\partial z} &= a_z \end{aligned}\right\} \qquad (3.35)$$

---

**EXAMPLE 3.10**

In steady incompressible flow the velocity components are given as follows

$$u = 2x \quad m/s$$
$$v = \tfrac{1}{2}y \quad m/s$$
$$w = -\frac{5}{2}z \quad m/s$$

Determine the pressure distribution in the flow if the pressure equals the atmospheric pressure at the point (0, 2, −1).

---

Data of the Problem

* steady, incompressible flow

$$* \quad u = 2x , \quad v = \tfrac{1}{2}y , \quad w = -\tfrac{5}{2} z$$

$$* \quad P = P_{atm} \quad \text{at } (x,y,z) = (o,2,-1)$$

Requirement

* The pressure distribution, P = P (x,y,z)

Solution

Assume a gravitational field with acceleration - g in the direction z. Euler's equation in the direction x, Eq. (3.35), yields

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

or

$$\frac{\partial p}{\partial x} = -\rho\,(2x \times 2) = -4\rho x$$

$$P = -2\rho x^2 + f_1 (y, z) \qquad \text{(I)}$$

Appling Euler's equation in the direction y, Eq. (3.35), gives

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \frac{1}{4} y$$

$$\frac{\partial p}{\partial y} = -\frac{\rho}{4} y$$

$$P = -\frac{\rho}{8} y^2 + f_2 (x,z) \qquad \text{(II)}$$

Appling Euler's equation in the direction z, Eq. (3.35), gives

$$-g - \frac{1}{\rho}\frac{\partial p}{\partial z} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{25}{4} z$$

or

$$\frac{\partial p}{\partial z} = -\rho g - \frac{25}{4}\rho z$$

$$P = -\rho g z - \frac{25}{8}\rho z^2 + f_3 (x,y) \qquad \text{(III)}$$

The three expressions for P as given by Eqs. (I), (II) and (III) are supposed to be identical since P should be the same as concluded by the three expressions. Comparison of the three expressions gives the values of $f_1$, $f_2$ and $f_3$. For example comparing (I) and (II) we get

$$P = -2\rho x^2 + f_1(y,z)$$
$$= -\frac{\rho}{8} y^2 + f_2(x,z)$$

Therefore,

$$f_1(y,z) = -\frac{\rho}{8} y^2 + \phi_1(z)$$
$$f_2(x,z) = -2\rho x^2 + \phi_2(z)$$

Now substituting by the value for $f_1$ (y,z) in Eq. (I) and compare the result to Eq. (III) we get

$$P = -2\rho x^2 - \frac{\rho}{8} y^2 + \phi_1(z)$$
$$= -\rho g z - \frac{25}{8}\rho z^2 + f_3(x,y) \qquad \text{(IV)}$$

By comparing the above two equations one may conclude that

$$\phi_1(z) = -\rho g z - \frac{25}{8}\rho z^2 + K \qquad \text{(V)}$$

where K is a constant. Then Eq. (IV) becomes

$$P = -2\rho x^2 - \frac{\rho}{8} y^2 - \rho g z - \frac{25}{8}\rho z^2 + K$$
$$= -\frac{\rho}{8}(16x^2 + y^2 + 8gz + 25z^2) + K \qquad \text{(VI)}$$

Since $P = P_{atm}$ at point (0,2, – 1), then

$$P_{atm} = -\frac{\rho}{8}(o + 4 - 8g + 25) + K$$

or

$$K = P_{atm} + \frac{\rho}{8}(29 - 8g) = P_{atm} + \frac{\rho}{8}(29 - 8 \times 9.8)$$
$$= P_{atm} - \frac{\rho}{8} \times 49.4$$

and Eq. (IV) becomes

$$P - P_{atm} = -\frac{\rho}{8}(16x^2 + y^2 + 8gz + 25z^2 + 49.4)$$

## 3 – 7 – Conservation of Energy

The previous analyses were made employing the laws of conservation of mass, momentum principle and Newton's second law. Further development of some of these laws shows that the law of conservation of energy has useful applications to fluids in motions. The conservation of energy implies that the energy content in a fluid element is constant. The following analyses are made under the conditions that the flow is steady and irrotational or steady and streamlined. The mathematical expression for the conservation of energy under the above conditions is known as Bernoulli's equation.

### 3 – 7 – 1 – Bernoulli's equation for steady irrotational flow

For a steady flow (that is independent of time) Euler's equations (Eq. 3.34) take the form

$$\beta_x - \frac{1}{\rho}\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \qquad (3.36a)$$

$$\beta_y - \frac{1}{\rho}\frac{\partial p}{\partial y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \qquad (3.36b)$$

$$\beta_z - \frac{1}{\rho}\frac{\partial p}{\partial z} = u\frac{\partial w}{\partial x} + u\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \qquad (3.36c)$$

The external force components $\beta_x$ , $\beta_y$ and $\beta_z$ may be expressed as partial derivatives of a particular potential (e.g. magnetic or gravitational field) denoted by $\Omega$, where

$$\beta_x = \frac{\partial \Omega}{\partial x} \quad , \qquad \beta_y = \frac{\partial \Omega}{\partial y} \qquad \text{and} \qquad \beta_z = \frac{\partial \Omega}{\partial z} \qquad (3.37)$$

Therefore Eqs. (3.36) become

$$\frac{\partial \Omega}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \qquad (3.38a)$$

$$\frac{\partial \Omega}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \qquad (3.38b)$$

$$\frac{\partial \Omega}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \qquad (3.38c)$$

Substituting the conditions of irrotationality as given by Eqs. (3.5) into the above equations we get

$$\frac{\partial \Omega}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} \qquad (3.39a)$$

$$\frac{\partial \Omega}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} = u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial y} \qquad (3.39b)$$

$$\frac{\partial \Omega}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} = u\frac{\partial u}{\partial z} + v\frac{\partial v}{\partial z} + w\frac{\partial w}{\partial z} \qquad (3.39c)$$

Multiplying Eqs. (3.39a), (3.39b) and (3.39c) by dx, dy and dz respectively and adding the three equations, we get

$$
\begin{aligned}
&\left( \frac{\partial \Omega}{\partial x} dx + \frac{\partial \Omega}{\partial y} dy + \frac{\partial \Omega}{\partial z} dz \right) \\
&\quad - \frac{1}{\rho} \left( \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz \right) \\
&= u \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz\right) \\
&\quad + v \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz\right) \\
&\quad + w \left(\frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz\right)
\end{aligned}
\tag{3.40}
$$

This may be simplified to

$$
\begin{aligned}
d\Omega - \frac{1}{\rho} dp &= u\, du + v\, dv + w\, dw \\
&= \tfrac{1}{2} d \left(u^2 + v^2 + w^2\right)
\end{aligned}
$$

or

$$
-d\Omega + \frac{dp}{\rho} + d\, \frac{C^2}{2} = 0 \tag{3.41}
$$

where the total velocity C is given as follows

$$
C^2 = u^2 + v^2 + w^2 \tag{3.42}
$$

In many engineering problems, the only external force that acts on the fluid may be that of the gravitational field. In such cases $d\Omega$ is given as follows

$$
\begin{aligned}
d\Omega &= \frac{\partial \Omega}{\partial z} dz = B_z\, dz , \\
&= -g\, dz
\end{aligned}
\tag{3.43}
$$

hence Eq. (3.41) becomes

$$
g\, dz + \frac{dp}{\rho} + C\, dC = 0 \tag{3.44}
$$

that is

$$
\int g dz + \int \frac{dp}{\rho} + \frac{C^2}{2} = \text{constant} \tag{3.45}
$$

The above equation is known as Bernoulli's equation.

The first term in the equation represents the potential energy per unit mass, the second term is the pressure energy per unit mass, and the third term is the kinetic energy per unit mass of the fluid. Therefore Bernoulli's equation, Eq. (3.45) states that the sum of the above three kinds of energy is constant at any point in the flow on condition that the flow is

1. irrotational
2. steady
3. non-viscous
4. subject to gravitational field only

If the fourth restriction is not satisfied, like the case of a flow subjected to a magnetic field as well as to the gravitational field, Eq. (3.41) should be used instead of Eq. (3.45). Equation (3.45) can be further simplified by the assumption of an incompressible flow and a constant value of g. This yields

$$gz + \frac{P}{\rho} + \frac{C^2}{2} = \text{constant} \tag{3.46}$$

which is known as Bernoulli's equation for incompressible flow.

It is important to note here that the constant on the right hand side of Eq. (3.46) is the same all over the field of a fluid continuum as long as the flow is incompressible and irrotational. This is different from what will be shown later as the constant of Bernoulli's equation for a stream tube or a streamline in an incompressible but not necessarily irrotational flow.

### 3 – 7 – 2 – Bernoulli's equation derived for steady streamlined flow

A streamline is a line drawn in the flow whose tangent gives the direction of the velocity at any instant of time. For steady flow, the shape of the streamline does not change with time.

A fixed control volume of height ds in the streamline direction and of a cross sectional area dA is shown in Fig. (3.7). The pressure, density and velocity at the centre of the control volume are taken as P, ρ and C respectively.

Now the momentum principle is to be applied to the shown control volume. Selecting s as a coordinate in the direction of the streamline, the momentum principle (Eq. 3.25b) takes the form

$$\dot{G}_{s,cv} = \dot{G}_{s,i} - \dot{G}_{s,o} + \Sigma F_s \tag{3.47}$$

The individual terms of the above equation are to be determined now. The sum of the external forces in direction s is given as follows

$$\Sigma F_s = (P - \frac{\partial p}{\partial s}\frac{ds}{2})\ dA - (P + \frac{\partial p}{\partial s}\frac{ds}{2})\ dA - s dA\ dsg\ \frac{dz}{ds}$$

$$= - \frac{\partial p}{\partial s} \, ds \, dA - \rho g \, dA \, ds \, \frac{dz}{ds} \qquad (3.48a)$$

By definition of a streamline there is no flow crossing the streamline. Therefore the momentum influx and efflux to and from the control volume respectively, become

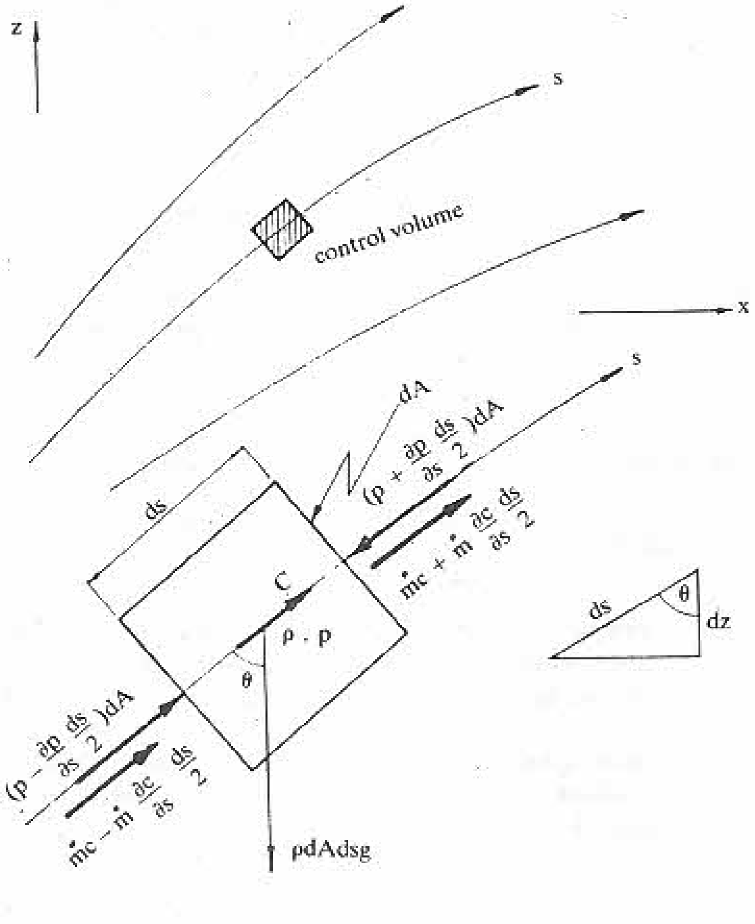



*Fig. 3.7 Control volume on a streamline*

$$\dot{G}_{s,i} = \dot{m}C - \dot{m}\frac{\partial C}{\partial s}\frac{ds}{2} \tag{3.48b}$$

$$\dot{G}_{s,o} = \dot{m}C + \dot{m}\frac{\partial C}{\partial s}\frac{ds}{2} \tag{3.48c}$$

where

$$\dot{m} = \rho\, c\, dA \tag{3.48d}$$

Also the rate of change of momentum of the control volume is given by the following relation

$$\begin{aligned} \dot{G}_{s,cv} &= \frac{\partial}{\partial t}(\rho\, C\, dA\, ds) \\ &= 0 \quad \text{(being a steady flow)} \end{aligned} \tag{3.48e}$$

Substituting by Eqs. (3.48) into Eq. (3.47) yields

$$\rho g \frac{dz}{ds} + \frac{\partial p}{\partial s} + \rho C \frac{\partial C}{\partial s} = 0 \tag{3.49}$$

This can be rewritten as follows

$$g \frac{dz}{ds} + \frac{1}{\rho}\frac{\partial p}{\partial s} + \frac{1}{2}\frac{\partial C^2}{\partial s} = 0 \tag{3.50}$$

Integrating the above equation along a streamline gives

$$gz + \int \frac{dP}{\rho} + \frac{C^2}{2} = \text{constant} \tag{3.51}$$

where the gravitational acceleration, $g$, is considered constant. In equation (3.51) the value of the constant changes from one streamline to another, and it is constant along a streamline. The application of Bernoulli's equation as given by Eq. (3.51), is restricted to the following conditions

1. Points on the same streamline,
2. Steady flow, and
3. non-viscous flow.

---

**EXAMPLE 3.11**

A water tank has an orifice of diameter 10 cm on its lower side. If the water level is kept constant at 4 meter above the orifice centreline. Determine the efflux velocity out of the orifice and the discharge.

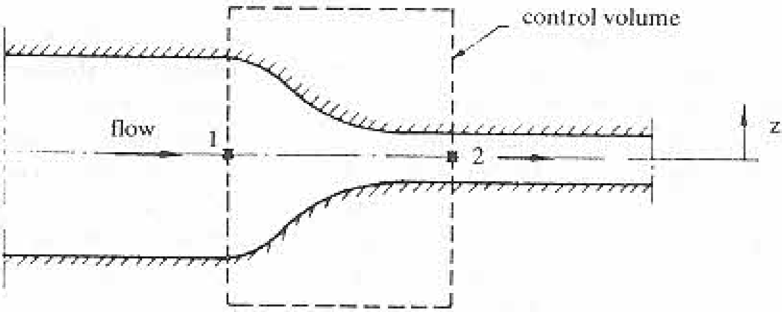
---

**EXAMPLE 3.12**

Air flows steadily and at low speed through a horizontal nozzle. At the nozzle inlet, the flow velocity, pressure and area are 10 m/s, 100 kN/m$^2$ and 0.1 m$^2$ respectively. At the nozzle exit, the area is 0.02 m$^2$. The flow is essentially incompressible and non-viscous. Determine the velocity and the pressure at the nozzle outlet. Take the density of air as 1.23 kg/m$^3$.

---

Problem Description




Data of the Problem

- Point 1: $C_1 = 10$ m/s , $P_1 = 10^5$ N/m$^2$, $A_1 = 0.1$ m$^2$
- Point 2 : $A_2 = 0.02$ m$^2$
- flow is steady, incompressible and non-viscous
- density of air, $\rho = 1.23$ kg/m$^3$

Requirements:

- $C_2$ and $P_2$

Solution

Apply the continuity equation, using the shown control volume

$$C_1 A_1 = C_2 A_2$$

i.e.

$$C_2 = C_1 \frac{A_1}{A_2} = 10 \times \frac{0.1}{0.02} = 50 \text{ m/s} \qquad \text{(I)}$$

Apply Bernoulli's equation, Eq. (3.51), along a streamline between sections (1) and (2) it yields

$$\frac{P_1}{\rho} + gz_1 + \frac{C_1^2}{2} = \frac{P_2}{\rho} + gz_2 + \frac{C_2^2}{2} \qquad \text{(II)}$$

Neglecting the potential difference between sections (1) and (2) Eq. (II) becomes

$$P_2 = P_1 + \frac{\rho}{2} ( C_1^2 - C_2^2 )$$
$$= 10^5 + \frac{1.23}{2} (100 - 2500)$$
$$P_2 = 98524 \quad N/m^2$$

---

## 3 – 8 – Calculation of the Pressure Distribution in a Flow of a Known Velocity Field

Both Euler's equations and Bernoulli's equation can be used to calculate the pressure distribution in a non-viscous flow if the velocity distribution is predetermined. Euler's equations are used to calculate the partial derivatives of the pressure with respect to the directions x,y and z. The pressure distribution is then determined by integrating those partial derivatives of the pressure. Whereas Bernoulli's equation gives directly the pressure distribution without extra work for integration. Therefore, since Bernoulli's equation is easier to apply one should be careful to assume all the conditions required for its application as given in subsection (3 – 6 – 1) and (3 – 6 – 2). If all the conditions are not satisfied there is no choice but to use Euler's equations. This will be illustrated by the following examples.

**EXAMPLE 3.13**

In EXAMPLE 3.10, is it possible to determine the pressure distribution for the flow by using Bernoulli's equation instead of Euler's equations? If yes, determine the pressure distribution.

---

Data of the Problem

* Steady, incompressible, non-viscous flow

* $u = 2x \;,\; v = \frac{1}{2}y \;,\; w = -\frac{5}{2} z$
* $p = p_{atm}$ at $(x, y, z) = (o, 2, -1)$

Requirements

* Check the possibility of using Bernoulli's equation to determine the pressure distribution.
* If possible, then determine the pressure distribution

Solution

Using Eqs. (3.4), the components of the angular velocity are determined as follows

$$\omega_x = \tfrac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = 0$$
$$\omega_y = \tfrac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial x}\right) = 0$$
$$\omega_z = \tfrac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

i.e. the flow is irrotational, and since it is steady Bernoulli's equation can be used to determine the pressure distribution in the flow. Under these conditions, Bernoulli's equation for incompressible flow in a gravitational field is given by Eq. (3.46) as follows

$$g\,z + \frac{P}{\rho} + \frac{C^2}{2} = K \qquad \text{(I)}$$

where K is a constant. Therefore,

$$P = -\rho\,\left(\tfrac{1}{2}\,C^2 + gz - K\right)$$

but

$$C^2 = u^2 + v^2 + w^2$$
$$= 4x^2 + \frac{1}{4}\,y^2 + \frac{25}{4}\,z^2$$

therefore

$$P = -\frac{\rho}{8}\,(16\,x^2 + y^2 + 25\,z^2 + 8gz - 8K) \qquad \text{(II)}$$

Since $P = P_{atm}$ at the point (o, 2, -1), then

$$P_{atm} = -\frac{\rho}{8}\,(o + 4 + 25 - 8g - 8K)$$

or,

$$\rho K = P_{atm} + \frac{\rho}{8}\,(29 - 8 \times 9.8)$$
$$= P_{atm} - \frac{\rho}{8}\,(49.4)$$

and Eq. (II) becomes

which is the same answer as that given in EXAMPLE 3.10. As we can see using Bernoulli's equation to obtain the pressure distribution is simpler than using Euler's equations. Note that if the restrictions to Bernoulli's equation do not apply there is no alternative but to use Euler's equations.

## 3 – 9 – Application of the Momentum Principle to Turbomachinery

The momentum principle is the basic principle in the operation of turbomachinery where the rate of change of the momentum of a fluid stream upon its reflection by a vane is equal to the force interacted between the fluid and the vane. This force when multiplied by the velocity of the vane gives the power exchanged between the fluid and the vane.

The analysis of a fluid jet reflected by a fixed vane is now discussed. This arrangement is shown in Fig. (3.8). Assume the following

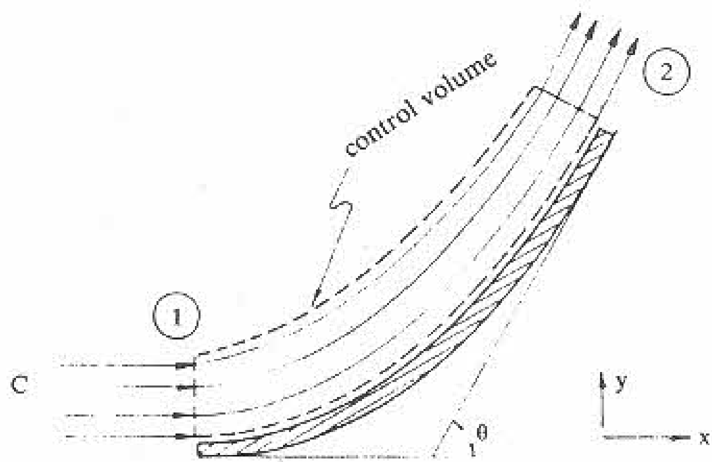



*Fig. 3.8 Flow of jet on a fixed vane*

1. steady, incompressible, non-viscous flow,
2. the change in the potential energy is negligible when compared with the kinetic energy of the jet and
3. the jet enters and leaves the vane tangentially.

Therefore, applying Bernoulli's equation between points (1) and (2) on the inlet and outlet of a streamline, as shown in Fig. (3. 8 ), yields

$$\frac{P_1}{\rho} + \frac{C_1^2}{2} = \frac{P_2}{\rho} + \frac{C_2^2}{2} \quad (3.52)$$

Assume $P_1 = P_2$ atmospheric pressure, the above equation becomes

$$C_1 = C_2 = C \quad (3.53)$$

Now let us apply the momentum principle, Eq. (3.27a), to the control volume shown in Fig. (3. 8 ). The following equation can be obtained

$$F_x = m_{cv}\, a_{x,cv} + u_{cv}\, \dot{m}_{cv} + \dot{m}c \cos\theta - \dot{m}c \tag{3.54}$$

where $F_x$ is the reaction of the vane on the control volume in direction x.

For a fixed vane each of the first two terms in the right hand side of the above equation equals zero. Thus, Eq. (3.54) is simplified to

$$F_x = \dot{m}\, C\, (\cos\theta - 1) \tag{3.55a}$$

Similarly the momentum principle in direction y, Eq. (3.27b) gives

$$F_y = \dot{m}\; C \sin\theta \tag{3.55b}$$

Again $F_y$ is the reaction of the vane on the control volume in direction y. Note that $F_x$ and $F_y$ are the components of the external force acting on the control volume due to the vane. This external force is equal in magnitude and direction to the force required to hold the vane at rest.

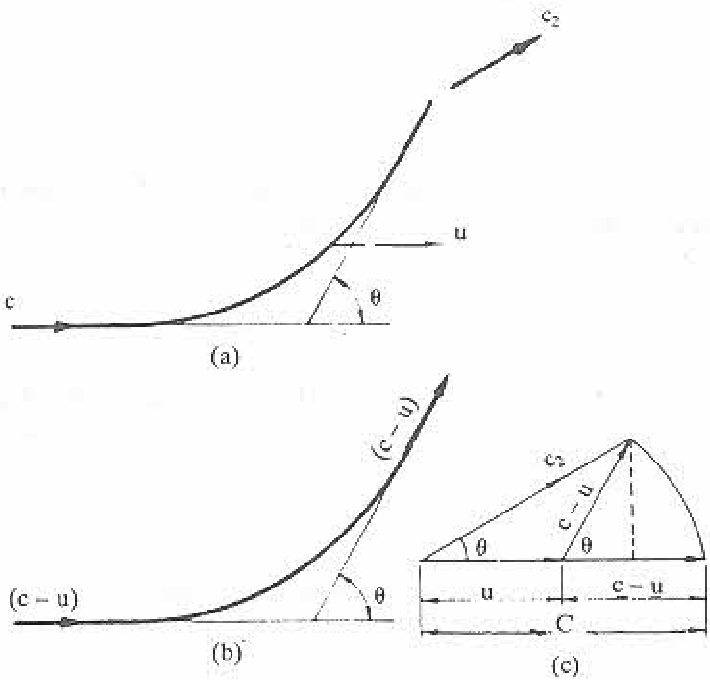



*Fig. 3.9 Flow of a jet on a moving vane*

Let us now consider a moving vane with a constant velocity ,u, as shown in Fig. (3.9a). If we superimpose a velocity $(-u)$ on both the fluid and the vane, the analysis becomes equal to that of a fixed vane as shown in Fig. (3.9b). This means that the flow enters tangential to the vane with a relative velocity (C-u) and leaves at a relative velocity (C-u) tangential to the vane. This is better illustrated in the polar vector diagram in Fig. (3.9c), from which the magnitude of the leaving absolute velocity $C_2$ is given as follows

$$C_2 = \sqrt{\{u + (C-u)\cos\theta\}^2 + \{(C-u)\sin\theta\}^2} \tag{3.56a}$$

and it makes an angle $\phi$ with direction x, that is given by the relation

$$\tan\phi = \frac{(C-u)\sin\theta}{u+(C-u)\cos\theta} \tag{3.56b}$$

The fixed vane shown in Fig. (3.9a) follows the same relations as that in Eq. (3.55) but with a fluid inlet velocity (C-u) instead of C. Therefore, the components of the reaction of the vane on the fluid control volume are

$$F_x = \dot{m}\,(C-u)\,(\cos\theta - 1) \tag{3.57a}$$

$$F_y = \dot{m}\,(C-u)\sin\theta \tag{3.57b}$$

The components of the action of the jet on the vane have the same magnitudes as the components $F_x$ and $F_y$ but in the opposite direction. When the vane moves with a constant velocity u in the direction x, the power transferred from the fluid to the vane is given as follows

$$\text{Power} = -F_x \cdot u \tag{3.58a}$$

$$= -\dot{m}u\,(C-u)\,(\cos\theta - 1) \tag{3.58b}$$

Another way to prove Eqs. (3.57a) and (3.57b) can be through direct application of the momentum principle Eqs. (3.27a) and (3.27b) to the moving vane shown in Fig. (3. 9 a). In direction x, this gives

$$F_x = \dot{m}\,C_2\cos\phi - \dot{m}\,C$$

From Fig. (3.9c), the value of $C_2\cos\phi$ can be substituted to give

$$F_x = \dot{m}\,\{u + (C-u)\cos\theta\} - \dot{m}\,C$$
$$= \dot{m}\,(C-u)\,(\cos\theta - 1)$$

which is the same as Eq. (3.57a). Similary, the momentum principle in direction y gives

$$F_y = \dot{m}\ C_2\ \sin\ \phi$$

and replacing $C_2 \sin \phi$ by its equivalent expression from Fig. (3.9c) yields

$$F_y = \dot{m}\ (C-u)\ \sin\ \theta$$

which is the same as Eq. (3.57b).

**EXAMPLE 3.14**

A 5 cm diameter nozzle is used to produce a water jet having a uniform velocity 50 m/s. The jet strikes tangentially a vane moving at a constant velocity 10 m/s. If the vane has a deflection angle of 60°, draw the polar vector diagram for the flow over the vane. Also, determine the absolute velocity of the fluid leaving the vane and the power transmitted to the vane.

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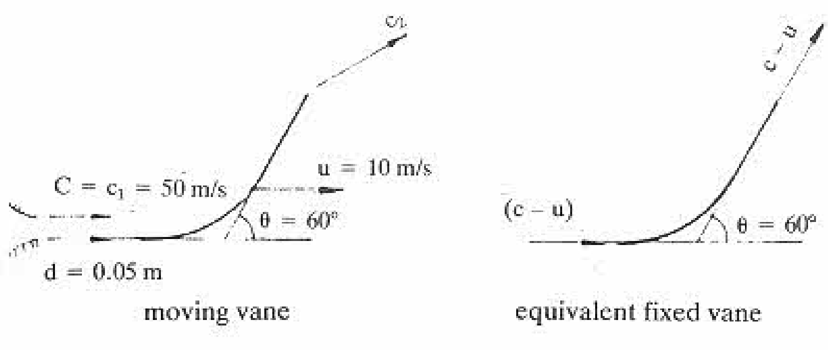
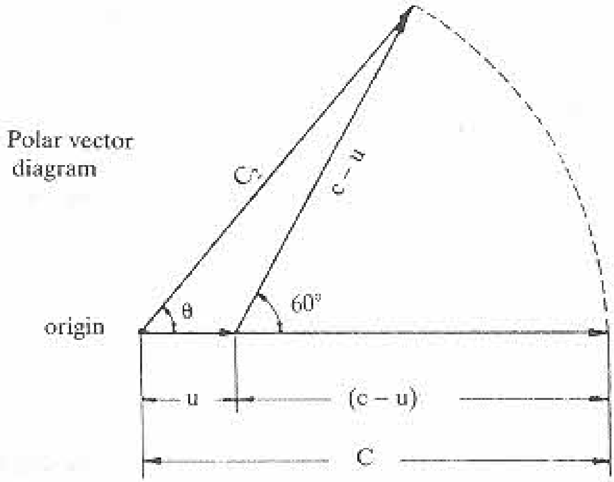
Problem Description




Data of the Problem
* water jet strikes a vane
* numerical data is shown in the problem description

Requirements
* polar vector diagram
* absolute velocity of the fluid leaving the vane, $C_2$
* the power transmitted to the vane

Solution




From the diagram,

$$C_2 = 46 \ \text{m/s}$$
$$\phi = 49.5^{\circ}$$

Also the value of $C_2$ and $\phi$ can be determined from Eqs. (3.56), these are respectively as follows

$$C_2 = \sqrt{(10 + 40 \cos 60)^2 + (40 \sin 60)^2} = 45.8 \ \text{m/s}$$

$$\phi = \tan^{-1} \left( \frac{40(\sin 60)}{10+40 \ (\cos 60)} \right) = 49.11^{\circ}$$

In order to calculate the power transmitted to the vane Eq. (2.58b) is used

$$\begin{aligned} \text{Power} &= - \dot{m} \ u \ (C-u) \ (\cos \theta - 1) \\ &= - \rho \ \frac{\pi}{4} \ d^2 \ u \ (C-u)^2 \ (\cos \theta - 1) \\ &= -10^3 \times \frac{\pi}{4} \times (0.05)^2 \times 40 \times 10 \times 40 \times (\cos 60 - 1) \\ &= 15707.96 \quad \text{Watt} \end{aligned}$$

(N.m/s = Watt)

## 3 – 10 – Static, Dynamic and Total Pressure in Incompressible Flow

### 3 – 10 – 1 – Definitions

For steady incompressible non-viscous flow with no significant change in the elevation z, Bernoulli's equation, Eq. (3.51) can be written as follows

$$\frac{P}{\rho} + \frac{C^2}{2} = \text{constant} \tag{3.59a}$$

At two points (1) and (2) along a streamline, the above equation yields

$$\frac{P_1}{\rho} + \frac{C_1^2}{2} = \frac{P_2}{\rho} + \frac{C_2^2}{2} = \text{constant} \tag{3.59b}$$

Now assume that at stage 2 the velocity of the fluid is zero, then we get

$$P_2 = P_o = P_1 + \frac{\rho C_1^2}{2} = P + \frac{\rho C^2}{2} \tag{3.60}$$

where P and C are the pressure and velocity at any point along the streamline and $P_o$ is the pressure at zero velocity which is called the stagnation pressure or the total pressure. The equation also shows that the stagnation pressure $P_o$ generally changes from one streamline to another. If the flow is also irrotational, then the stagnation pressure will be the same through out the whole flow regiem.

Equation (3.60) lends useful means for determining the velocity at a point in a flow field. For example, consider a fluid travelling at velocity "C" and pressure "P" in a duct or otherwise. If by some means a small amount of this fluid is brought to stand still, the pressure of this amount can be given by equation (3.60), which may be rewritten as follows

$$P_o = P + P_c \tag{3.61}$$

where P is the static pressure and $P_c$ is the dynamic pressure, which is defined by the following equation

$$P_c = \frac{\rho C^2}{2} = P_o - P \tag{3.62}$$

3 – 10 – 2 – **Pitot static tube**

In practice, the apparatus used for measuring the velocity of a fluid by employing the above described phenomenon is known as the static pitot tube. A sketch of this tube is shown in Fig. (3.10). At point (1) a flowing fluid has pressure P and velocity C. At point (2) (i.e. tube nose), a stagnation state exists with stagnation pressure $P_o$ which can be measured with a manometer. However, when the flow turns around the tube at tube nose, it moves parallel to the walls of the tube with pressure P and velocity C almost equal to those of the main stream. A manometer connected to the holes at location (3) measures the static pressure at that location.

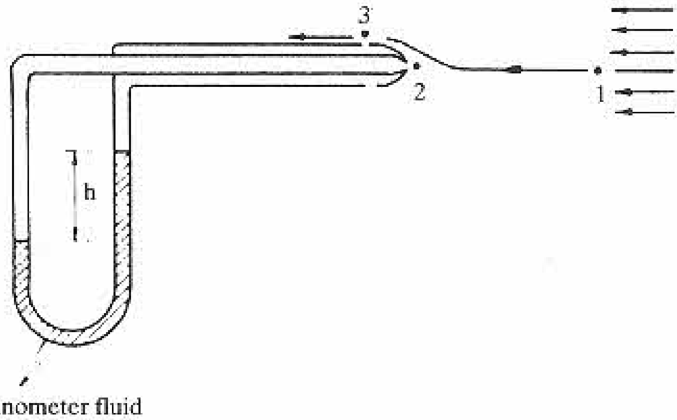



*Fig. 3.10 Pitot static tube*

A differential manometer connected to holes (2) and (3) measures the difference between $P_o$ and P representing the dynamic pressure of the flow, which is given by Eq. (3.62). Thus

$$\frac{\rho C^2}{2} = P_o - P = gh\ (\rho_m - \rho) \quad (3.63a)$$

where $\rho_m$ is the density of the manometer liquid. Equation (3.63a) can also give the velocity C in the form

$$C = \sqrt{2gh\ (\frac{\rho_m}{\rho} - 1)} \quad (3.63b)$$

**EXAMPLE 3.15**

A pitot tube is used to measure the velocity of a stream of oil of specific gravity 0.9. If the attached differential water manometer shows a reading of 30 cm, what is the velocity of the oil?

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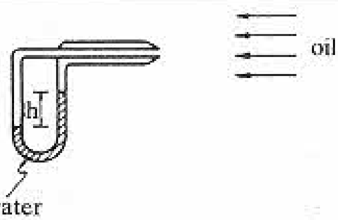
Problem Description



Data of the Problem

* flow of oil, $\rho = 0.9 \rho_W$
* manometer fluid is water, $\rho_m = \rho_\omega = 1000 \text{ kg/m}^3$
* $h = 0.3$m of water

Requirements

* velocity of oil, C

Solution

Applying Eq. (3.63b) yields

$$C = \sqrt{2gh\left(\frac{\rho_m}{\rho} - 1\right)} = \sqrt{2 \times 9.8 \times 0.3 \times \left(\frac{1}{0.9} - 1\right)}$$
$$= 0.81 \text{ m/s}$$

---

## PROBLEMS ON CHAPTER THREE

### Problems on Section 3 – 1 to 3 – 4

3.1. The velocity field in a non-viscous fluid flow is given by

$u = yz + t$ , $v = xz - t$ , $w = xy$

( i ) Find the acceleration components of a fluid element in terms of x,y,z and t.

(ii ) Find the total velocity and total acceleration of a fluid element at a point with coordinate (2,1,2) at the end of the first unit of time.

(iii) Is the flow irrotational?

3.2. The velocity distribution of a turbulent flow inside a circular pipe is given in terms of centerline velocity $C_{max}$, the pipe radius R, and the distance from the centerline r as follows

$$\frac{C}{C_{max}} = \left(1 - \frac{r}{R}\right)^{\frac{1}{n}}$$

where n is a constant that depends on the flow Reynolds number. Express the volume flow rate in the pipe in terms of n, $C_{max}$ and R.

3.3. A water channel of width 2b and depth H. If the velocity of water (m/s) at any location is given as follows

$$\frac{C}{20} = \left\{ 1 - \left( \frac{y}{b} - 1\right)^2 \right\} \left( 1 - \frac{z}{H} \right)$$

where y and z are the horizontal distances from the wall and the water depth from the surface respectively. Find the expression for the mass flow rate in the channel. For a channel with width of 50 m and depth of 10 m determine the value of the mass flow rate.

3.4. In Fig. (3.11), water flows in a pipeline reducer at rate of 3000 kg/s. Calculate the mean velocity in the 300 mm and 200 mm pipes.

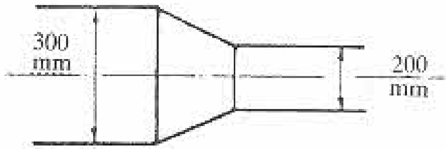



*Fig. 3.11*

**Problems on Section 3 – 5**

3.5. Water flows in a pipe of inlet and exit cross sectional areas of 0.3 $m^2$ and 0.15 $m^2$ respectively. The velocity of water at the inlet is 1.8 m/s. If the velocity is assumed uniform and normal to the pipe cross section, find the velocity of the water at the pipe exit.

3.6. Air flows steadily through a compressor at the rate of 50 kg/min entering with a pressure of 100 kN/$m^2$ and temperature 20°C and leaving with a pressure of 900 kN/$m^2$ and a temperature of 200°C. The velocity in both the intake and delivery pipes is at an average of 30 m/s. Determine the diameters of intake and delivery pipes ($R_{air}$ = 287 J/kg K).

3.7. The velocity of an incompressible fluid flow at the inlet of a triangular cross section bend is constant. The dimensions of the cross section vary between inlet and exit as shown in Fig. (3.12). If the velocity distribution at the exit is linear as depicted from the figure, calculate the velocity at the inlet of the bend.

3.8. At the exit of an open rectangular channel bend the velocity at the water surface varies linearly with the width from 4 m/s at one side to 1 m/s at the other side and varies with the depth so that the magnitude at the free surface is twice that at the base. The width of the channel is 1 m and the depth of water in the channel is 1.5m. Calculate the mass flow rate in the channel.

3.9. A rigid vessel containing 2 $m^3$ of a perfect gas at a pressure of 80 kN/$m^2$ is connected to a pump which extracts the gas at a constant rate of 0.2 $m^3$/min, discharging it to atmosphere. The temperature of the gas in the vessel may be assumed to remain constant during the pumping process. Find the time taken to reduce the pressure in the vessel to 13 kN/$m^2$.

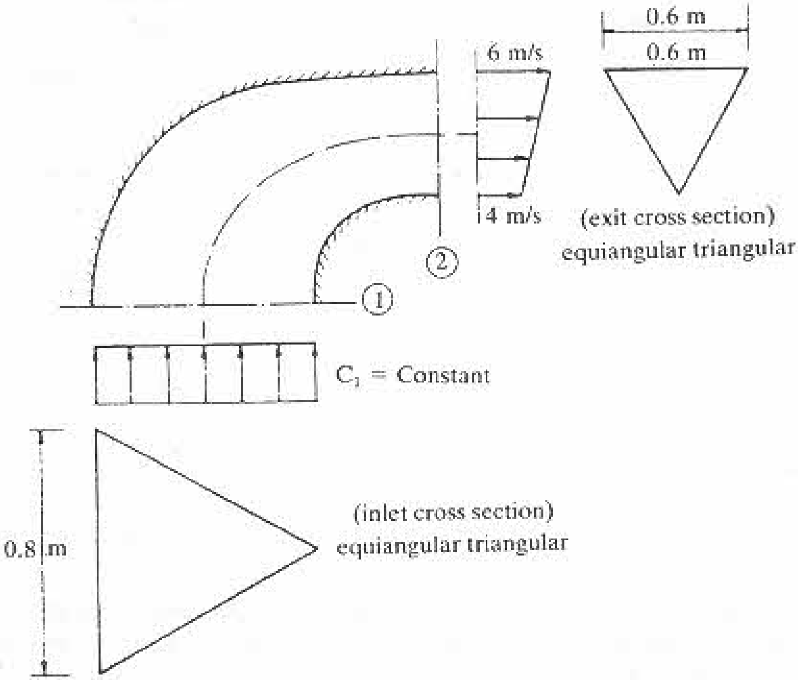



*Fig. 3.12*

3.10. The following are two velocity components for incompressible flow. Find the third velocity component and determine in each case whether the flow is rotational or not.

a) $u = x^2 + y^2 + z^2$ , $v = -xy - yz - xz$

b) $u = \ln(y^2 + z^2)$ , $v = \sin(x^2 + z^2)$

c) $u = x + y + z$ , $v = x^2 + y^2 + z^2$

3.11. In Fig. (3.13), water issues upward from a nozzle at the center of a cylindrical cavity with dimensions as indicated on the figure. The velocity distribution at the nozzle is given as follows

$$C = 10 \{ 1 - (\frac{y}{R_N})^2 \} \text{ m/s}$$

where y is the radius from the centerline of the nozzle, $R_N$ is the radius of the nozzle. Calculate the average velocity at the exit from the annular space.

3.12. In the device shown in Fig. (3.14) calculate $C_4$ if the flow is incompressible and steady.

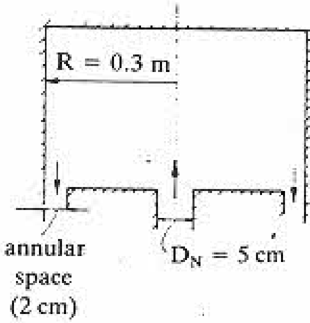



*Fig. 3.13*

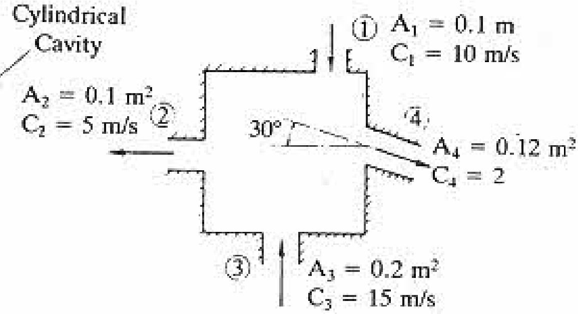



*Fig. 3.14*

3.13. In an automobile factory, an air compressor of a rated output about 20 kg/s is used to blow tires to a gauge pressure of 2 atmosphere. Calculate the number of tires per hour to be blown by the compressor. You may assume that each tire has a constant volume of 0.05 $m^3$ during the blowing process which is assumed isothermal. Consider air as a perfect gas (R = 287 J/kg N), and the atmospheric conditions as $P = 10^5$ N/m² and T = 30°C.

3.14. Derive the continuity equation for a cylindrical infinitesimal control volume (cylindrical coordinates).

3.15. In a compressible unidirectional steady flow, the velocity component in the direction x is given by

$$u = 5 + 4x^2y + z^2$$

Determine the density distribution in the field.

3.16. A water tank of cross sectional area A is used to supply water to a four stories building at an average rate of $\dot{m}$ kg/s. The tank has a hole where water leaks at a rate $\Delta \dot{m}$ kg/s which is proportional to the square root of the water height in the tank. The tank has a float that operates a pump to feed water to the tank when the water height inside the tank becomes lower than $H_1$ meter. The float shuts the pump off when the water height exceeds $H_2$ meter. If the pump supplies water at a constant rate of $\dot{M}$ kg/s which is larger than $(\dot{m} + \Delta \dot{m})$, calculate the operation time for the pump and the time interval between operation.

3.17. Calculate the mass flow rate of exhaust gases from a car that consumes about 5 kg/hr of fuel if the airfuel ratio is about 15:1.

3.18. A section of pipe carrying water contains an expansion chamber with a free surface whose area is 2 $m^2$ (Fig. 3.15). The inlet and outlet pipes are both 1 $m^2$ in area. At a given instant, the average velocity at section 1 is 3 m/s into the chamber. Water flows out at section 2 with an average velocity of 4 m/s. Find the rate of change of free surface level at the given instant in m/s. Indicate whether the level rises or falls.

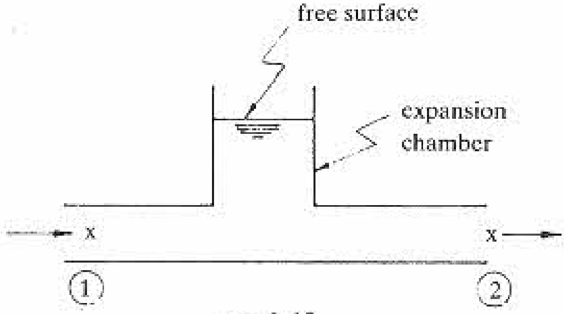



*Fig. 3.15*

3.19. In ancient Egypt, circular vessels filled with water sometimes were used as crude clocks. The vessels were shaped in such a way that, as water drained from the bottom, the surface level dropped at a constant rate h. Assume that water drains from a hole of area A. The water speed leaving the vessel is approximated by $C = \sqrt{2gh}$, where h is the height of the liquid free surface above the jet exit. Find an expression for the radius of the vessel, r, as a function of the height, h. Determine the volume of water needed so that the clock will operate for n hours.

**Problems on Section 3 – 6**

3.20. Calculate the force required to hold a garden hose of nozzle with area ratio 2:1 and with inlet diameter of 5 cm. The water pressure at the inlet is $2 \times 10^5 N/m^2$ at mass flow rate of 15 kg/s.

3.21. A tower is to be built near the shore of a sea. The construction is expected to extend to about 5 meters under the water surface and 15 meters above it. The velocity of the water at the tower location changes linearly with the depth z from the surface according to the relation.

$$C = 15 - 0.5 z \text{ m/s}$$

where z in meters. If the tower has a square cross section with side length of 4 meters, determine the maximum force on any of the side walls of the tower and its acting centre.

3.22. The velocity of air at the inlet and outlet of a triangular bend is as shown in Fig. (3.16). Calculate the force exerted on the bend. Neglect the weight of air in the bend and assume incompressible flow with density $\rho = 1.25 \text{ kg/m}^3$.

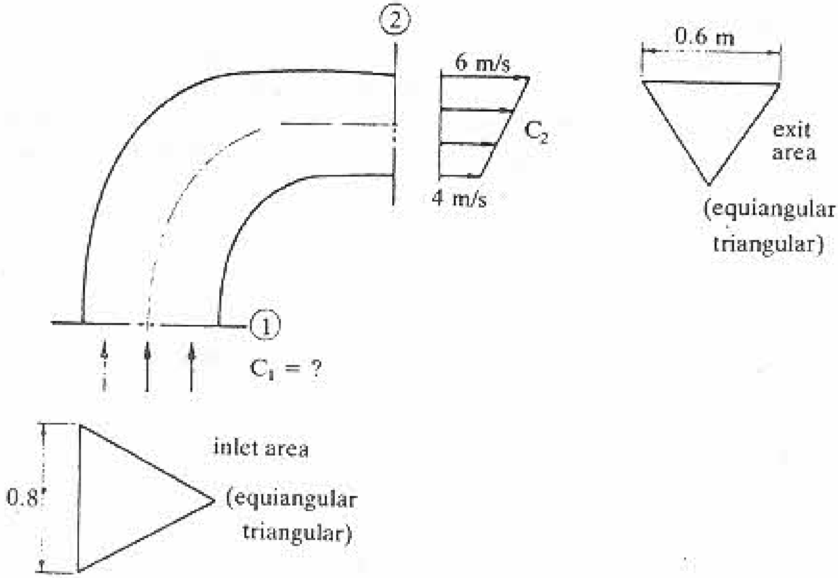



*Fig. 3.16*

3.23. The pressure distribution of an inviscid flow in a gravitational field is given as follows

$$p = 3\,x^2\,yt + 4y + 5t^2$$

calculate the components of the instantaneous local acceleration in the directions x, y and z.

3.24. Determine the external force (per unit mass of the fluid) acting on a steady incompressible flow with the following characteristics

$$u = 1 + y\;,\; v = w = o$$

$$P = \rho\,(2x^2 + y + 9.8z)$$

where $\rho$ is the fluid density.

**Problems on Section 3 – 7**

3.25. Which of the following motions are kinematically possible for a steady incompressible flow? For possible motions determine whether or not Bernoulli's equation can be applied.

| | | | |
|---|---|---|---|
| (a) | $u = kx$ | $v = -ky$ | $w = 0$ |
| (b) | $u = kx$ | $v = -ky$ | $w = kx$ |

(c) $u = kx$ $v = -ky$ $w = kz$
(d) $u = kx$ $v = ky$ $w = kz$
(e) $u = kx$ $v = ky$ $w = -2kz$
(f) $u = \frac{kx}{(x^2 + y^2)}$ $v = \frac{ky}{(x^2 + y^2)}$ $w = 0$

3.26. A round water tank (D = 1 m) contains water to a height of 5 m from its base. The tank is open to the atmosphere. Calculate the time required to drain the tank through 2 cm hole in its base.

3.27. Water flows vertically downward from a tap into the atmosphere. If the stream diameter and its velocity at the tap outlet are 15 mm and 0.3 m/s respectively, determine the velocity and the stream diameter at 0.6m below the tap.

3.28. A 20 cm water pipe has in it a venturi meter of throat diameter 12.5 cm, which is connected to a mercury manometer showing a difference of 87.8 cm. Find the velocity in the throat and the discharge. If the upstream pressure is 690 kN/m$^2$, what power would be given up by the water if it was allowed to discharge to atmospheric pressure.

3.29. Find the time to empty a tank 6m$^2$ cross-sectional area and 2 m deep through a 20cm diameter flared orifice which is 1 m below the tank bottom. Consider the orifice to have a coefficient of discharge equal to 0.62.

3.30. A vertical pipe 300 mm diameter conveys water at rates up to 100 kg/s. The flow is metered by a venturi, the pressure difference between the inlet and the throat being measured by a U-tube containing mercury. If the maximum manometer reading is not to exceed 600 mm mercury, determine the minimum permissible throat diameter of the venturi. Assume the venturi to have a coefficient of discharge $C_d = 0.97$.

3.31. An aerofoil is so shaped that the velocities along the upper and lower surfaces are respectively 25 percent greater than, and 25 percent smaller than, the velocity of the oncoming stream. What is the lift force on such a wing, 15 m long and 3 m chord, at 320 km/hr.

3.32. Air flows steadily from a large pipe to atmosphere through a 50 mm diameter nozzle. Velocity of air in the large pipe is negligible. Atmospheric pressure is 100 kN/m$^2$ and the pressure in the large pipe is 150 kN/m$^2$ absolute. Atmospheric temperature is 17°C, and the relationship between pressure and density everywhere in the flow is $P/\rho^{1.4}$ = constant. Considering air as an ideal gas of which R = 0.287 kJ/kg K, determine the mass flow rate through the nozzle.

3.33. Water coming from a single reservoir flows first in a main pipe which branches after that to two pipes in a vertical plane. Water discharges into atmosphere at a velocity of 3.5 m/s from one branch at a point 2 m higher than the centreline of the main pipe, and from the other branch at some unknown velocity but at a point 3.5m lower than the centre line of the main pipe. Each pipe has a cross-sectional area of 0.1 $m^2$. Assuming ideal flow, find the pressure and the volume flow rate in the main pipe.

**Problems on Section 3 – 8**

3.34. The velocity components of an incompressible flow field are given as follows

$$u = yzt \; , \; v = xzt \; , \; w = xyt$$

Find the expression for the pressure at any point in terms of x, y, z and t.

**Problems on Section 3 – 9**

3.35. Find the force exerted on a fixed vane when a jet of water with discharging rate of 0.2 $m^3/s$ at speed of 50 m/s is reflected through 135° angle.

3.36. A water jet having a mass flow rate 100 kg/s at a velocity 30 m/s strikes a vane moving at a velocity of 10 m/s. The vane has a reflection angle of 60°. Determine the power transmitted to the vane and the absolute velocity leaving the vane.

3.37. Determine the horse power per vane that can be obtained from a series of vanes curved through 170° moving with velocity 60 m/s. Draw the polar diagram and calculate the energy remaining in the jet. Also, verify the energy balance on the jet. Note that the water jet comes tangential to the vane inlet and its velocity and cross sectional area are 120 m/s and 0.002 $m^2$, respectively.

3.38. A jet of oil (sp. gr. 0.8) flowing at a rate of 20 liters/min from a nozzle of 2.5 cm diameter impinges on a plate inclined at 60° to horizontal. What is the force acting on the plate and what are the flows along the surface of the plate.

**General Problems on Chapter 3**

3.39. Water flows from a constant head tank through an orifice on its base of diameter d = 0.05 m. The jet from the orifice impinges on a horizontal plate at about 0.5m below the orifice, as shown in Fig. (3.17). Find the magnitude and direction of the net force exerted by the jet on the plate.

3.40. A liquid jet is issuing upward against a flat board of weight W and supporting it as indicated in Fig. (3.18). Determine the equilibrium height of the board above the nozzle exit as a function of nozzle area and the nozzle exit velocity.

3.41. A pitot-static tube is carefully aligned with an air stream of density 1.23kg/$m^3$. If the attached differential manometer shows a reading of 150 mm of water, what is the velocity of the air stream?

3.42. Water flows under a sluice gate on a horizontal bed at the inlet to a flume.

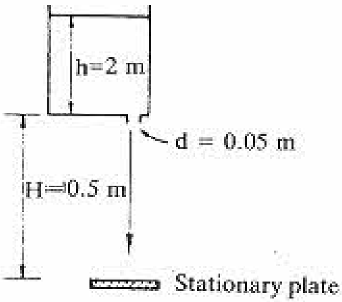



*Fig. 3.17*

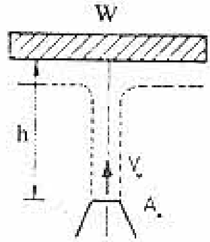



*Fig. 3.18*

Above the gate, the water level is 50 cm and the velocity is negligible. At the vena contracta below the gate, the flow stream lines are straight and the depth is 5 cm. A uniform flow and a negligible friction may be assumed. Determine the flow velocity downstream from the gate and the discharge per meter of width.

3.43. A stand-pipe filled with fluid has openings at the quarter lengths, as shown in Fig. (3.19). Assuming that the fluid level remains constant, calculate the points at which the jets of fluid will strike the ground at the level of the base of the stand pipe.

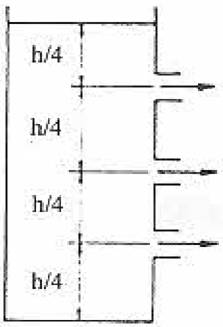



*Fig. 3.19*

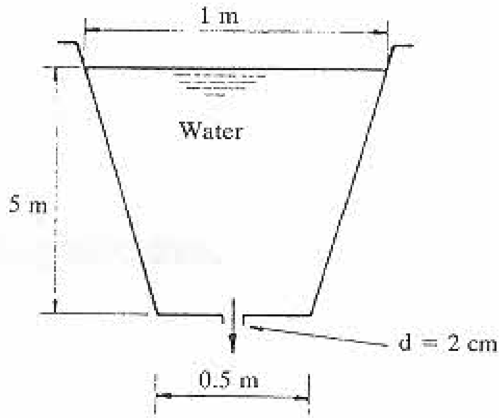



*Fig. 3.20*

3.44. Determine the time required to empty the tank having a conical shape as indicated in Fig. (3.20).

3.45. A water jet issues from a nozzle (2.5 cm in diameter) at a rate of 0.2 $m^3$/min. The jet impinges on a curved vane with an exit angle of 45°. A force transducer measures the reaction force which is found to be 30 N in the direction of the jet. Determine the absolute velocity of the vane.

CHAPTER FOUR

# VISCOUS FLOW

## 4 – 1 – Introduction

In the previous chapter it was assumed that the fluid is ideal, i.e. there is no internal friction between the flowing fluid layers and when this fluid flows over a solid wall no fluid particles adhere to the wall. The assumption of ideal fluid is used to simplify the governing equations of the fluid flow. This has proved to be useful in many practical applications. A wide range of practical problems in fluid mechanics seems to be better explained if the concept of real fluid is introduced. A real fluid has the following two important characteristics. First it deforms continuously under the application of any shearing force. Second, flowing particles of such a fluid would adhere to any adjacent wall. This characteristic is known as the no-slip condition. Both characteristics are related to what is called the viscosity of the fluid. An ideal flow, therefore, assumes zero viscosity.

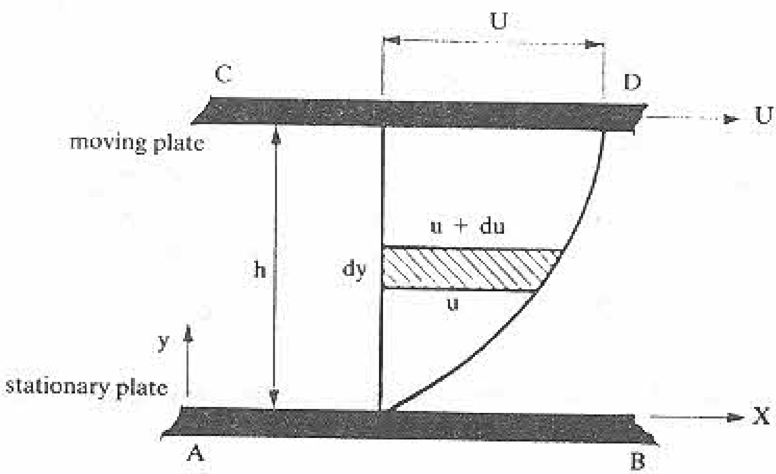



*Fig. 4.1 Motion of fluid between two plates*

In Fig. (4.1), AB represents a fixed plane and CD a parallel plane at a distance h from AB. The space between the planes is filled with a fluid and the upper plane CD moves uniformly with velocity U parallel to plane AB. According to the no slip condition the fluid particles adjacent to plane AB have zero velocity while those adjacent to the plane CD move with velocity U. Fluid particles between the two planes have velocities that range from zero to U. This gradual change of the velocity of the fluid particles is caused by the viscosity of the fluid which is a measure of the internal friction between the fluid layers.

At steady state conditions it is found that there is a shearing stress $\tau$ at any fluid layer and its magnitude is proportional to the velocity gradient across the layer, that is

$$\tau \propto \frac{\partial u}{\partial y} \tag{4.1}$$

or

$$\tau = \mu \frac{\partial u}{\partial y} \tag{4.2}$$

where $\mu$ is a proportionality constant known as the coefficient of viscosity, the dynamic viscosity or simply the viscosity. A fluid for which the coefficient of viscosity does not depend on the shear or the velocity gradient is known as Newtonian fluid.

Experiment shows that the viscosity of a given fluid depends in general on its temperature and pressure. The viscosity of liquids usually decreases with rise of temperature and is nearly independent of pressure for pressures not exceeding a few atmospheres. The viscosity of gases increases with rise of temperature and is independent of pressure, except when the latter is very high.

The SI unit of the dynamic viscosity $\mu$ is the Pascal. second (Pa. s). The ratio of the dynamic viscosity $\mu$ to the fluid density $\rho$ is known as the kinematic viscosity $\nu$ and its SI unit is $m^2/s$.

The shear stress $\tau$ in Eq. (4.2) is defined as the shear force divided by the area parallel to the force. This shear stress $\tau$ is positive if it acts on a positive area and it is negative if it acts on a negative area. An area is considered positive if the normal out from it is positive.

## 4 – 2 – Laminar and Turbulent Flows

Viscous flow patterns are classified according to their internal structure to either laminar or turbulent flow. Laminar flow is the kind where the fluid flows in laminae or layers. Turbulent flow is a random motion. Unlike laminar flow, turbulent flow invariably has three-dimensional vorticity fluctuations. This property makes the diffusivity of matter in turbulent flow much higher than that of laminar flow. This causes rapid mixing in turbulent flow as compared to the mixing in laminar flow. For example when dye is injected in a laminar flow, the dye colours a streamline and remains as an identifiable layer for a while, whereas in case of turbulent flow the dye is quickly tangled colouring the whole flow. The determination of whether the flow is laminar or turbulent depends on many factors among which is the value of the Reynolds number*. Turbulent flow always occurs at high Reynolds number relative to that of laminar flow. The value of the critical Reynolds number at which transition takes place from laminar to turbulent depends on the configuration of the flow. For example the critical Reynolds number for a flow in a pipe is equal to 2300 while that for a flow over a flat plate is about $3.2 \times 10^5$.

## 4–3– Boundary Layer Concept

When a flow of uniform velocity flows over a solid surface, such as a flat plate, the fluid particles in contact with the surface remain at zero velocity due to the no slip condition at the wall. Ahead of the leading edge of a plate the flow comes with a uniform velocity $U_\infty$ (see Fig. 4.2) and the layer adjacent to the plate is suddenly brought to rest. The change in the velocity of fluid from $U_\infty$ to zero at the leading edge occurs abruptly. The thickness of the fluid layer over which this change of velocity takes place is zero. By flowing further down stream on the plate more fluid particles that become adjacent to the plate are brought to zero velocity. Because of the internal friction of fluid, the retardation effect initiated at the wall penetrates to the other layers of the flow causing a gradual change of velocity from zero to $U_\infty$. The region of the flow where this velocity gradient occurs is called the boundary layer. The flow inside the boundary layer having a distinct velocity gradient due to viscosity is called viscous flow, while the flow outside the boundary layer having nearly a constant velocity of $U_\infty$ that is not affected by viscosity may be treated as non-viscous flow.

The thickness $\delta$ of the boundary layer is defined as the normal distance to the surface at which the flow velocity becomes about 99% of that of the main flow. In Fig. (4.2) the velocity profiles at three different stations across the boundary layer are illustrated. In each profile the boundary layer thickness $\delta$ is indicated. Another basic boundary layer thickness is the displacement thickness $\delta_d$ (see Fig. 4.3) which refers to the displacement of the main flow due to slowing down of fluid particles in the boundary layer zone. Referring to the figure the quantity of mass flow displaced as a result of the existence of the boundary layer is

$$\int_0^\delta \rho \, (U_\infty - u) \, dy$$

(*) Reynolds number is a non-dimensional parameter relating viscosity $\mu$, density $\rho$, velocity C and a characteristic length of flow L so that $Re = \rho CL/\mu$

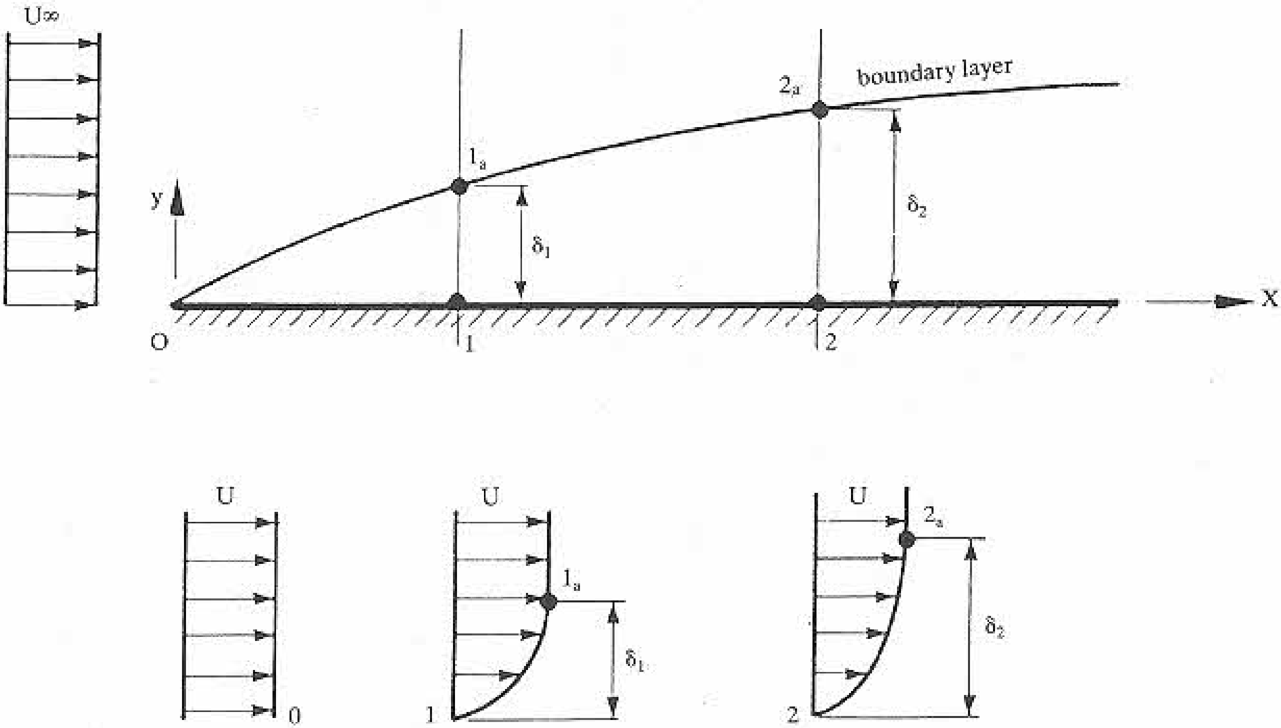



*Fig. 4.2 Boundary layer growth*

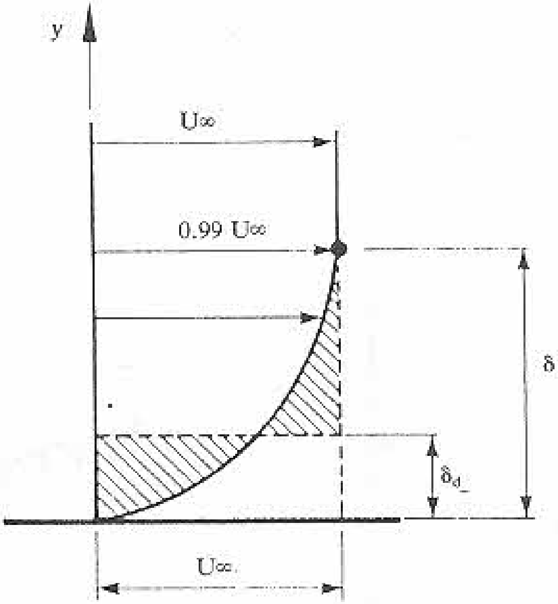



*Fig. 4.3 Velocity profile inside boundary layer*

Such quantity, of thickness $\delta_d$, is displaced to the main flow where the velocity is $U_\infty$, thus

$$\rho\, U_\infty\, \delta_d = \int_0^\delta \rho\,(U_\infty - u)\, dy$$

The above equation gives the definition of the displacement thickness $\delta_d$ and for incompressible flow it gives

$$\delta_d = \int_0^\delta \left( 1 - \frac{u}{U_\infty} \right) dy \tag{4.3}$$

A third boundary layer thickness that is known as the momentum thickness, $\delta_m$, is defined as follows

$$\delta_m = \int_0^\delta \frac{u}{U_\infty} \left( 1 - \frac{u}{U_\infty} \right) dy \tag{4.4}$$

---

**EXAMPLE 4.1**

Find the relation between the displacement thickness $\delta_d$ and boundary layer thickness $\delta$ of an incompressible flow over a flat plate if the velocity distribution inside the boundary layer is given by the following expression

$$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

where U is the velocity far from the plate and y is the normal distance measured from the plate.

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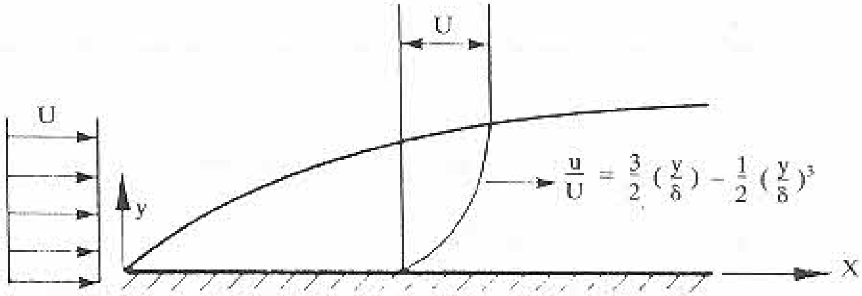
Problem Description




Data of the Problem

* Incompressible flow over a flat plate
* Configuration as given in Problem Description

Requirement

* The relation between $\delta_d$ and $\delta$

Solution

From Eq. (4.3) one may write

$$\delta_d = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

$$= \int_0^\delta \left\{1 - \frac{3}{2}\frac{y}{\delta} + \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right\} dy$$

$$= \left(y - \frac{3}{4}\frac{y^2}{\delta} + \frac{1}{8}\frac{y^4}{\delta^3}\right)_0^\delta$$

$$= \delta - \frac{3}{4}\delta + \frac{1}{8}\delta = \frac{3}{8}\delta$$

i.e.

$$\delta_d = \frac{3}{8}\delta$$

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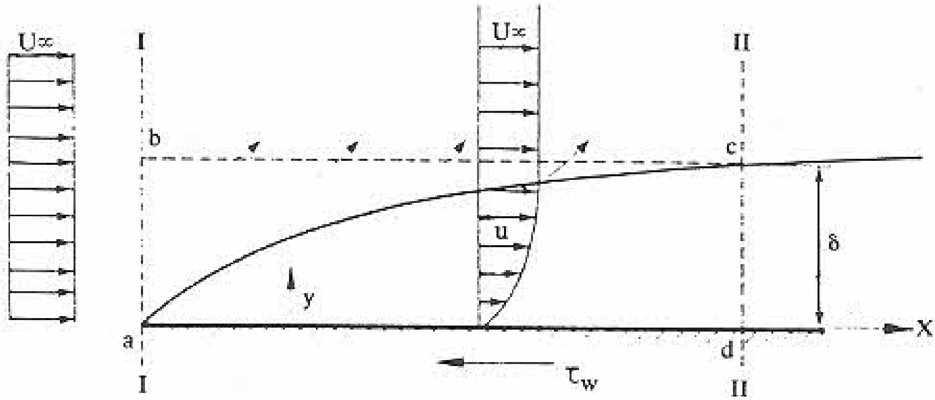


*Fig. 4.4 Boundary layer flow over a flat plate*

## 4 – 4 – Boundary Layer Over a Flat Plate

Consider a flat plate set in a parallel incompressible flow of constant velocity $U_\infty$ in the direction of the plate. Let x be the coordinate along the plate and y is the coordinate perpendicular to the plate as shown in Fig. (4.4). Also, let I—I be a section taken at the leading edge of the plate, while II—II is another section at a distance x from the leading edge where the boundary layer thickness is $\delta$. Assuming a unit width for the plate, then the mass flow rates through (ab) and (cd), respectively, are

$$ {}_a\dot{m}_b = \rho\, U_\infty\, \delta \tag{4.5}$$

and

$$ {}_c\dot{m}_d = \int_0^\delta \rho\, u\, dy \tag{4.6}$$

Applying the conservation of mass on the control volume (abcd) gives

$$ {}_b\dot{m}_c = {}_a\dot{m}_b - {}_c\dot{m}_d $$

or

$$ {}_b\dot{m}_c = \rho \int_0^\delta (U_\infty - u)\, dy \tag{4.7}$$

Also, applying the momentum principle in direction x on the control volume (abcd) gives

$$ \Sigma F_x = \dot{G}_{x,0} - \dot{G}_{x,i} \tag{4.8}$$

The values of the individual terms of Eq. (4.8) are

$$\Sigma F_x = -\int_0^x \tau_w \cdot dx \tag{4.9}$$

$$\dot{G}_{x,i} = \dot{G}_{x,ab} = \rho U_\infty^2 \delta \tag{4.10}$$

$$\dot{G}_{x,0} = \dot{G}_{x,bc} + \dot{G}_{x,cd}$$

$$= \rho \int_0^\delta (U_\infty - u)\, dy + \int_0^\delta \rho u^2 dy \tag{4.11}$$

Therefore Eq. (4.8) becomes

$$-\int_0^x \tau_w dx = \rho U \int_0^\delta (U_\infty - u)\, dy + \int_0^\delta \rho u^2 dy - \rho U_\infty^2 \delta$$

or

$$\int_0^x \tau_w \, dx = \rho \int_0^\delta u\, (U_\infty - u)\, dy \tag{4.12}$$

The above equation may be rewritten as follows

$$\mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \rho U_\infty^2 \frac{\partial}{\partial x} \int_0^\delta \frac{u}{U_\infty} \left(1 - \frac{u}{U_\infty}\right) dy \tag{4.13}$$

Now assume that

$$\frac{u}{U_\infty} = F(\eta) \tag{4.14}$$

where

$$\eta = \frac{y}{\delta} \tag{4.15}$$

Then, Eq. (4.13) becomes

$$\frac{\mu}{\delta} \left.\frac{\partial F}{\partial \eta}\right|_{\eta=0} = \rho U_\infty \frac{\partial \delta}{\partial x} \int_0^1 F(1 - F)\, d\eta \tag{4.16}$$

Also, assume

$$F(\eta) = \frac{3}{2}\eta - \frac{\eta^3}{2} \tag{4.17}$$

which satisfies the boundary conditions $u=0$ at $y=0$ and $u=U_\infty$ at $y=\delta$. Therefore, substitution of Eq. (4.17) in Eq. (4.16) gives

$$\frac{3}{2}\ \frac{\mu}{\rho\,\delta\,U_\infty} = \frac{\partial\delta}{\partial x}\int_0^1 \left(\frac{3}{2}\eta - \frac{\eta^3}{2}\right)\left(1-\frac{3}{2}\eta + \frac{\eta^3}{2}\right)d\eta$$

$$\frac{3}{2}\ \frac{\mu}{\rho\delta U_\infty} = 0.1393\ \frac{d\delta}{dx}$$

i.e.

$$\delta d\delta = 10.7681\ \frac{\mu}{\rho U_\infty}\ dx$$

By integration over the length x of the plate gives

$$\int_0^\delta \delta d\delta = 10.7681\ \frac{\mu}{\rho U_\infty}\int_0^x dx$$

or

$$\frac{\delta^2}{x^2} = 21.5363\ \frac{\mu}{\rho x U_\infty}$$

i.e.

$$\frac{\delta}{x} = \frac{4.64}{\sqrt{Re_x}} \tag{4.18}$$

where $Re_x$ is the Reynolds number of the free stream based on the distance x. Equation (4.18) applies for laminar flow only. For turbulent flow the experiments showed that.

$$\frac{\delta}{x} = \frac{0.376}{(Re_x)^{0.2}} \tag{4.19}$$

The shear stress, $\tau_w$, is now calculated making use of Eq. (4.18). This yields

$$\tau_w = \mu\ \frac{\partial u}{\partial y}\bigg|_{y=0}$$

$$= \mu \frac{U_\infty}{\delta} \left. \frac{\partial F}{\partial \eta} \right|_{\eta=0}$$

$$= \frac{3}{2} \mu U_\infty \cdot \frac{\sqrt{Re_x}}{4.64\ x}$$

or

$$\tau_w = C_f \left( \frac{1}{2} \rho U_\infty^2 \right) \tag{4.20a}$$

where

$$C_f = \frac{0.646}{\sqrt{Re_x}} \tag{4.20b}$$

The parameter $C_f$ is known as the skin friction coefficient. Integrating Eq. (4.20a) over the length L of the plate, the mean shear stress, $\tau_w$, then becomes

$$\tau_w = \frac{1}{L} \int_0^L \tau_w \, dx$$

$$= \frac{1}{L} \int_0^L \frac{0.646}{\sqrt{Re_x}} \left( \frac{1}{2} \rho U_\infty^2 \right) dx$$

$$= \frac{1.292}{\sqrt{Re_L}} \left( \frac{1}{2} \rho U_\infty^2 \right) \tag{4.21a}$$

More accurate velocity distribution than the one given by Eq. (4.17) showed that for laminar flow, the mean shear stress is given by

$$\tau_w = \frac{1.328}{\sqrt{Re_L}} \left( \frac{1}{2} \rho U_\infty^2 \right) \tag{4.21b}$$

or

$$\left. \begin{aligned} \tau_w &= C_D \left( \frac{1}{2} \rho U_\infty^2 \right) \\ C_D &= \frac{1.328}{\sqrt{Re_L}} \end{aligned} \right\} \tag{4.22}$$

In the previous section analysis was carried out for laminar flow to predict the variation of the boundary layer thickness with the distance along the plate and to predict the local shear stress and the average shear stress on a plate of length L. Similar analysis can not be performed for transition and turbulent flows. Instead, one may rely on experimental results which gave the following correlations (Ref: Schlichting, 1968, p. 601).

transition: $$C_D = \frac{0.074}{Re_L^{1/5}} - \frac{A}{Re_L} \quad , \; 5 \times 10^5 < Re_L \leq 10^7 \tag{4.23}$$

turbulent: $$C_D = \frac{0.074}{Re_L^{1/5}} \quad , \quad Re_L \geq 10^7 \tag{4.24}$$

where A is a constant having the following values

$$\left.\begin{array}{ll} A = 1700 & \text{at } Re_L = 5 \times 10^5 \\ A = 3300 & \text{at } Re_L = 10^6 \\ A = 8700 & \text{at } Re_L = 3 \times 10^6 \end{array}\right\} \tag{4.25}$$

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**EXAMPLE 4.2**

Wind is blowing with an average velocity of 60 km/hr over a flat plate. Calculate the boundary layer thickness at a point on the plate about 50 cm from the leading edge. Note that for air $\rho = 1.23$ kg/m³ and $\mu = 18.76 \times 10^{-6}$ Pa.s.

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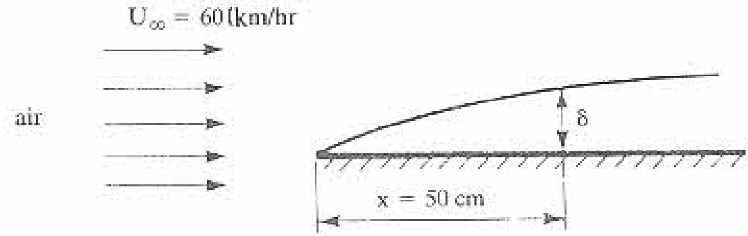
Problem Description




Data of the Problem

* as shown in Problem Description
* $\rho_{air} = 1.23$ kg/m³, $\mu_{air} = 18.76 \times 10^{-6}$ Pa.s

Requirements

* thickness of the boundary layer $\delta$ at x = 0.5 m from the plate leading edge.

Solution

$$U_\infty = 60 \times \frac{1000}{3600} = 16.67 \quad m/s$$

$$Re_x = \frac{\rho U_\infty x}{\mu} = \frac{1.23 \times 16.67 \times 0.5}{18.76 \times 10^{-6}}$$

$$= 5.46 \times 10^5 > 3.2 \times 10^5 \quad \text{i.e. turbulent flow}$$

$$\frac{\delta}{x} = \frac{0.376}{(Re_x)^{0.2}}$$

$$\delta = \frac{0.376 \times 0.5}{(5.46 \times 10^5)^{0.2}} = 0.0134 \quad m$$

$$\delta = 1.34 \quad cm$$

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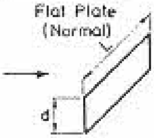
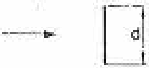
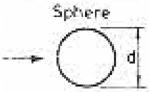
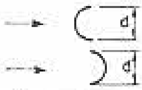
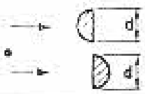
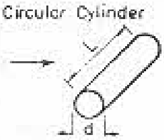
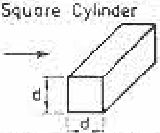
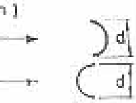
## 4 – 5 – Lift and Drag Forces

In the previous section a drag coefficient was defined for flow over a flat plate. This drag coefficient is used to determine the net force exerted by a flow parallel to a flat plate. Generally speaking all bodies immersed in fluid flow are exposed to a force due to both friction between flow and the surface and the pressure of the flow on the surface of the body. This force has two components: one component is in the direction of the motion and known as the drag force and the other component is perpendicular to the motion and known as the lift force. The drag force due to friction is called the skin friction drag while the drag force due to pressure distribution is called the pressure drag. When the flow is parallel to the surface of the body the drag will be a skin friction drag only. In terms of the dynamic pressure of the free stream ($1/2\ \rho\ U_\infty^2$ ) the drag and lift forces are respectively defined as follows

$$F_D = C_D\ A\ (\ \frac{1}{2}\ \rho\ U_\infty^2\ ) \tag{4.26}$$

$$F_L = C_L\ A\ (\ \frac{1}{2}\ \rho\ U_\infty^2\ ) \tag{4.27}$$

Table 4.1 Drag coefficient ($C_D$) for different geometries*

| Object | $C_D$ | Reynolds No. Range | Characteristic Length | Characteristic Area |
|---|---|---|---|---|
| Flat Plate (Tangential)   | $1.33\,(Re)^{-1/2}$ | laminar | L | Plate surface area |
| | $0.074\,(Re)^{-1/5}$ | $Re < 10^7$ | | |
| Flat Plate (Normal)   | L/d<br>1 1.18<br>5 1.2<br>10 1.3<br>20 1.5<br>30 1.6<br>∞ 1.95 | $Re > 10^3$ | d | Plate surface area |
| Circular Disk (Normal)   | 1.17 | $Re > 10^3$ | d | |
| Sphere   | $24(Re)^{1/2}$ | $Re < 1$ | d | Projected area |
| | 0.47 | $10^3 < Re < 3 \times 10^5$ | | |
| | 0.2 | $Re > 3 \times 10^5$ | | |
| Hollow Hemisphere   | 0.34 | $10^4 < Re < 10^6$ | d | Projected area |
| | 1.42 | $10^4 < Re < 10^6$ | | |
| Solid Hemisphere   | 0.42 | $10^4 < Re < 10^6$ | d | Projected area |
| | 1.17 | $10^4 < Re < 10^6$ | | |
| Circular Cylinder   | L/d<br>1 0.63<br>5 0.8<br>10 0.83<br>20 0.93<br>30 1.0<br>∞ 1.2 | $10^3 < Re < 10^5$ | d | Projected area |
| Square Cylinder   | 2.0 | $3.5 \times 10^4$ | d | Projected area |
| Square cylinders   | 1.6 | $10^4$ to $10^5$ | d | Projected area |
| Semitubular (infinete length)   | 2.3 | $4 \times 10^4$ | d | Projected area |
| | 1.12 | $4 \times 10^4$ | d | Projected area |

* With some modifications from: Hyges, W.F. and J.A. Brighton, Fluid *Dyhamics*, Shaum's Outline Series, McGraw-Hill Book Company, 1967.

where A is a characteristic area and $C_D$ and $C_L$ are called the drag and lift coefficients respectively. The characteristic area A is commonly taken as the surface area or the projected area normal to the flow direction. The values of $C_D$ and $C_L$ depend on the geometry of the body, flow configuration and Reynolds number. Values of the drag coefficient $C_D$ for different geometries are tabulated in Table (4.1).

**EXAMPLE 4.3**

The following objects are immersed in a steady incompressible flow continuum of water of 20 cm/s velocity and 20°C temperature:

a) flat plate parallel to flow with 50 cm length and 20 cm width,

b) flat plate normal to flow with 2 m height and 10 m width,

c) sphere of diameter 5 cm, and

d) circular cylinder normal to the flow of 10 cm diameter and 150 cm length.

Determine the drag force for these objects.

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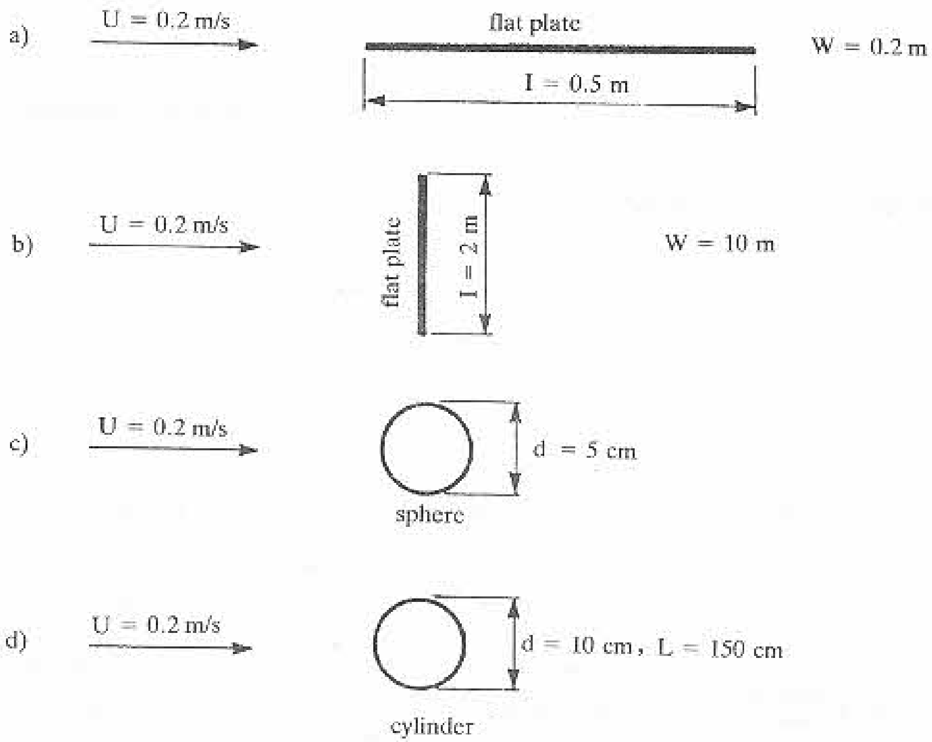
Problem Description



Data of the Problem

* Steady incompressible flow
* Water (at 20°C) flowing with velocity 0.2 m/s
* Objects in the flow are as shown in Problem Description

Requirement

* Drag force on each of the objects

Solution

The density and dynamic viscosity of water at 20°C are determined from Table (4.2) as follows

$$\rho = 1001 \ kg/m^3$$
$$\mu = 10.046 \times 10^{-4} \ Ns/m^2$$

In the following Table the drag force on each object is calculated by employing the following

$$Re_\ell = \frac{\rho U \ell}{\mu}$$

where $\ell$ is the characteristic length as indicated in Table 4.1. Also, A is the characteristic area as indicated in Table 4.1.

The drag force, then becomes

$$F_D = C_D \ A \ (\tfrac{1}{2} \rho U^2)$$

| case | object | $\ell$ | Re | A | $C_D$ | $F_D$ |
|---|---|---|---|---|---|---|
| a | flat plate (tnagential) | 0.5 m | $9.96\times10^4$ | $\ell\times W = 0.3\ m^2$ | $4.21\times10^{-3}$ | 0.00084 N |
| b | flat plate (normal) | 2 m | $3.99\times10^5$ | $\ell\times W = 20\ m^2$ | 1.2 | 480 N |
| c | sphere | 0.05 m | $9.96\times10^3$ | $\frac{\pi}{4} d^2 = 1.96\times10^{-3} m^2$ | 0.47 | 0.0184 N |
| d | cylinder (normal) | 0.1 m | $1.99\times10^4$ | $dL = 0.15\ m^2$ | 0.88 | 2.64 N |

### 4 – 6 – Fully Developed Flow

In section 4.4 boundary layer flow over an external surface such as a flat plate was considered. It was shown that the thickness of the boundary layer grows with the distance down stream. If one now considers internal flow inside a conduit, such as a pipe, the growth of the thickness of the boundary layer will of course be limited by the dimensions of the conduit.

In Fig. (4.5) we consider internal flow in a conduit with characteristic cross sectional dimension $\ell$ (for a pipe the diameter D). The flow approaches the conduit with a velocity $U_\infty$. Because of the no slip condition at the walls of the conduit, a boundary layer flow will be initiated at the entrance. As it is expected this boundary layer will grow in thickness until it reaches its possible maximum thickness at the centerline of the conduit. The distance between the entrance and the point of maximum thickness of the boundary layer is known as the entrance length, $L_e$ . This entrance length depends on the nature of the flow as being laminar or turbulent. For laminar flow, experiments showed that the relation for $L_e$ takes the form

$$\frac{L_e}{D} \propto Re \qquad (4.28)$$

where Re is the Reynolds number $(Re = \frac{\rho \bar{C} \ell}{\mu})$ based on the average flow velocity $\bar{C}$ inside the conduit.

In the entrance region, between sections I and III the flow may be divided into two regions: boundary layer flow (viscous flow) near the walls and potential flow (non-viscous flow) otherwise. At every cross section between sections I and III the flow velocity changes from a value of zero at the wall to its maximum value $U_{max}$ between the edges of the boundary layer at the same cross section. The maximum velocity $U_{max}$ in the potential flow region increases with the distance downstream in order to satisfy the continuity equation. This is illustrated by the velocity profiles shown in Fig. (4.5). After the entrance region, i.e. after section III, the velocity profile takes a constant shape that is independent of the distance down stream. The flow after section III is known as fully developed flow (see Fig. 4.5).

In the following analysis fully developed laminar flow is to be considered in two configurations. In the first configuration flow between two plates (with upper plate moving) is analyzed. The second configuration is that of a fully developed flow inside a pipe.

#### 4 – 6 – 1 – Laminar flow between two parallel plates with upper plate moving at a constant velocity

Consider the flow between two horizontal parallel infinite plates as shown in Fig.

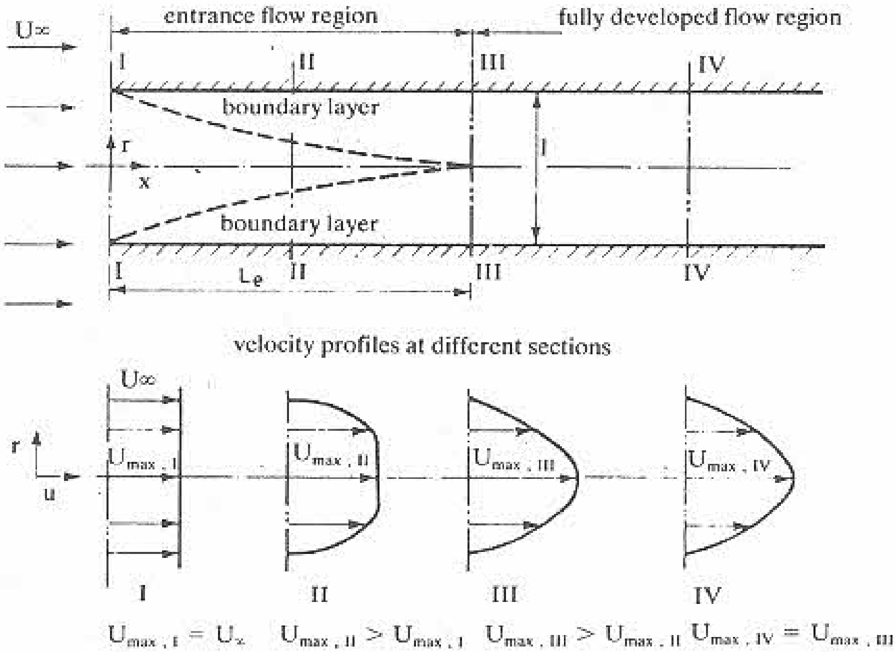



*Fig. 4.5 Entrance and fully developed flow regions in a conduit*

(4.6). The lower plate is stationary while the upper plate is moving horizontaly at a constant velocity U. The distance between the two plates is h. Assume:

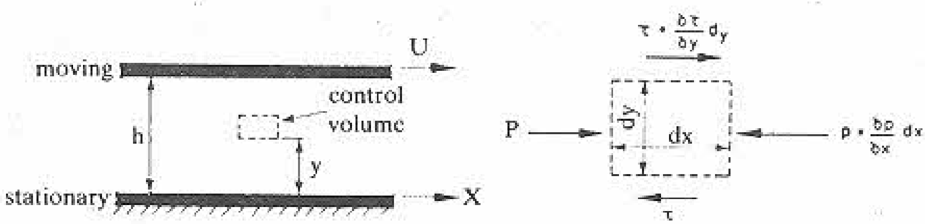



*Fig. 4.6 Laminar fully developed flow between two plates*

i) unidirectional flow, i.e. $v=0$ and $w=0$; ii) fully developed flow, i.e. $u=u\ (y)$; iii) steady flow, i.e. all quantities are independant of time; and iv) incompressible flow, i.e. $\rho=$constant. Let us now apply the momentum principle in the x – direction,

$$\dot{G}_{x,cv} = \dot{G}_{x,i} - \dot{G}_{x,o} + \Sigma F_x \tag{4.29}$$

on the infinitesmal control volume of unit width. The velocity u in the direction x at a fixed distance y is constant for a fully developed laminar flow. Thus:

$$\dot{G}_{x,i} = \int_{A_i} u_i \, dm_i = G_{x,o} = \int_{A_o} u_o \, d\dot{m}_o \tag{4.30}$$

i.e.

$$G_{x,i} = G_{x,o} \tag{4.31}$$

and since the control volume is stationary we conclude that

$$\dot{G}_{x,cv} = 0 \tag{4.32}$$

Therefore

$$\Sigma F_x = 0 \tag{4.33}$$

i.e.

$$- \frac{\partial p}{\partial x} dx\, dy + \frac{\partial \tau}{\partial y} dy\, dx = 0 \tag{4.34}$$

or

$$\frac{\partial \tau}{\partial y} = \frac{\partial p}{\partial x} \tag{4.35}$$

Also, since the flow is in the x direction no shear stress exists in the y direction. In addition, a fully developed flow assumes that $u = u\ (y)$ and the shear stress $\tau$ for a fluid with constant viscosity $\mu$ becomes

$$\tau = \mu \frac{\partial u}{\partial y} = \mu \frac{du}{dy} \tag{4.36}$$

Substitution by Eq. (4.36) into Eq. (4.35) gives

$$\mu \frac{d^2 u}{dy^2} = \frac{\partial p}{\partial x} \tag{4.37}$$

The above equation is integrated to the following

$$u = \frac{1}{2\mu} \frac{\partial p}{\partial x} y^2 + a_1 y + a_2 \tag{4.38}$$

where $a_1$ and $a_2$ are the integration constants. They are determined from the boundary conditions. The boundary conditions are

$$\left.\begin{array}{ll} \text{at } y = 0 & u = 0 \\ \text{at } y = h & u = U \end{array}\right\} \tag{4.39}$$

Therefore the values of $a_1$ and $a_2$ become

$$\left.\begin{array}{l} a_1 = \frac{U}{h} - \frac{h}{2\mu}\frac{\partial p}{\partial x} \\ a_2 = 0 \end{array}\right\} \tag{4.40}$$

and Eq. (4.36) is rewitten as follows

$$u = U\left(\frac{y}{h}\right) + \frac{h^2}{2\mu}\frac{\partial p}{\partial x}\left[\left(\frac{y}{h}\right)^2 - \left(\frac{y}{h}\right)\right] \tag{4.41}$$

The above equation estimates the velocity u in terms of the normal distance measured from the stationary plate (lower plate) y, the distance between the two plates h, the pressure gradient along the channel $\partial p/\partial x$ on the dynamic viscosity of the fluid $\mu$. The velocity distributions between the two plates are plotted in Fig. (4.7) for both positive and negative pressure gradients.

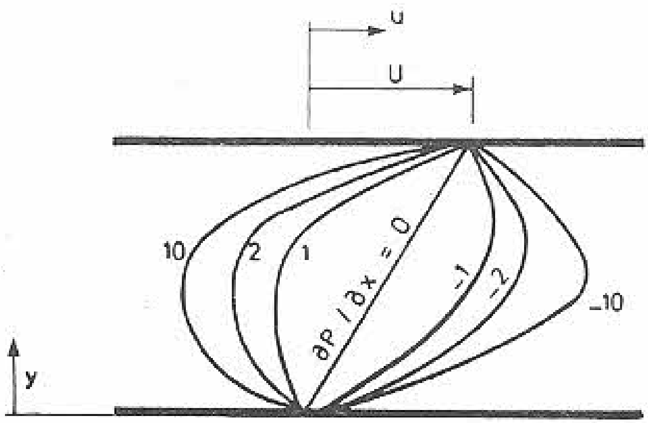



*Fig. 4.7 Velocity distribution of laminar flow between two plates, upper plate is moving with velocity U and lower plate is stationary*

The velocity distribution between two stationary plates (i.e. $U=0$) can easily be obtained from Eq. (4.41).

The shear stress $\tau$ is determined by substituting for u from Eq. (4.41) into Eq. (4.36). This yields

$$\tau = \frac{h}{2} \frac{\partial p}{\partial x} \left[ 2 \frac{y}{h} - 1 \right] + \mu \frac{U}{h} \tag{4.42}$$

The volume flow rate per unit width of the plate is given as follows

$$\dot{V} = \int_0^h u \, dy = \int_0^h U \frac{y}{h} dy + \int_0^h \frac{h^2}{2\mu} \frac{\partial p}{\partial x} \left( \frac{y^2}{h^2} - \frac{y}{h} \right) dy$$

$$= \frac{Uh}{2} + \frac{h^2}{2\mu} \frac{\partial p}{\partial x} \left( \frac{h}{3} - \frac{h}{2} \right)$$

$$= \frac{Uh}{2} - \frac{h^3}{12\mu} \frac{\partial p}{\partial x} \tag{4.43}$$

The average velocity of the flow is calculated by dividing the volume flow rate by the flow area, thus

$$\bar{c} = \frac{\dot{V}}{h} = \frac{U}{2} - \frac{h^2}{12\mu} \frac{\partial p}{\partial x} \tag{4.44}$$

The location of maximum velocity is determined by equating $\frac{du}{dy}$ (i.e. shear stress) to zero. This gives

$$\mu \frac{U}{h} = \frac{h}{2} \frac{\partial p}{\partial x} \left( 1 - 2 \frac{y}{h} \right)$$

i.e.

$$y_{max} = \frac{h}{2} - \mu \frac{U}{h} \cdot \frac{1}{\partial p / \partial x} \tag{4.45}$$

where $y_{max}$ is the distance from the stationary plate to the location of the maximum velocity.

### 4-6-2- Laminar flow in a pipe

In the present subsection fully developed laminar flow in a pipe is considered. For the pipe flow shown in Fig. (4.8) the following will be assumed:

1. unidirectional flow, i.e. one component velocity motion in x direction;

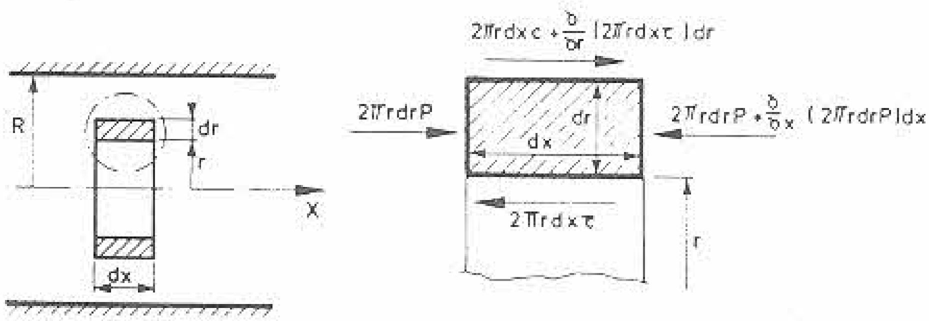
2. fully developed, i.e. $u = u(r)$;

3. steady flow, i.e. all quantities are independent of time;

4. axisymmetric flow, i.e. all quantities are symmetric around the centerline which means independent of the angle $\theta$ around the centerline; and

5. incompressible flow, $\rho$ = constant.

Select the cylindrical control volume of length dx and thickness dr as illustrated in the figure. The external forces are only the shear stress and the pressures. Since the flow is in the x direction only, no shear stress exists in the radial direction. Thus at the inner surface of the control volume the shear force is $2\pi r dx\ \tau$ and its direction




a. location of control volume

b. external forces on the control volume

*Fig. 4.8 Fully developed laminar flow in a pipe*

is in the negative direction of x because the surface is negative (i.e. the normal to the surface is in the direction of r). On the other hand the force on the outer surface is in the positive direction of x. The pressure forces on the control volume can also be determined as shown in Fig. (4.8). Applying the momentum principle in x direction gives

$$+\frac{\partial}{\partial r}(2\pi r\tau)\,dxdr - \frac{\partial}{\partial x}(2\pi r p)\,drdx = 0$$

or

$$\frac{\partial}{\partial r}(r\tau) = r\frac{\partial p}{\partial x} \tag{4.46}$$

substituting by $\tau = \mu\frac{du}{dr}$, the above equation becomes

$$\mu\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = r\frac{\partial p}{\partial x} \tag{4.47}$$

Since the left hand side of the above equation is a function of r only, the partial derivatives can be made full, i.e.

$$\mu \frac{d}{dr} \left( r \frac{du}{dr} \right) = r \frac{\partial p}{\partial x}$$

Integration with respect to r yields

$$u = \frac{1}{4\mu} \frac{\partial p}{\partial x} r^2 + a_1 \ln r + a_2 \tag{4.48}$$

where $a_1$ and $a_2$ are constants of integration. Two boundary conditions are necessary to determine the values of $a_1$ and $a_2$. These are

$$\begin{aligned} &\text{at } r = 0 \text{ , } u = \text{finite} \\ &\text{at } r = R \text{ , } u = 0 \text{ (no slip condition)} \end{aligned} \tag{4.49}$$

Substitution of Eqs. (4.49) into (4.48) yields the values of $a_1$ and $a_2$. Therefore, Eq. (4.48) becomes

$$u = - \frac{R^2}{4\mu} \frac{\partial p}{\partial x} \left( 1 - \frac{r^2}{R^2} \right) \tag{4.50}$$

The shear stress, therefore, becomes

$$\tau = \mu \frac{du}{dr}$$

$$\tau = \frac{r}{2\mu} \frac{\partial p}{\partial x} \tag{4.51}$$

The volume flow rate in the pipe is thus given by

$$\dot{V} = \int_o^R u . 2 \pi r \, dr$$

$$= - \frac{\pi R^2}{2\mu} \frac{dp}{dx} \int_o^R r \left( 1 - \frac{r^2}{R^2} \right) dr$$

$$\dot{V} = - \frac{\pi R^4}{8\mu} \frac{\partial p}{\partial x}$$

$$\dot{V} = - \frac{\pi R^4}{8\mu} \frac{\Delta p}{\Delta x} \tag{4.52}$$

where the pressure is assumed to be linear. The average velocity $\bar{u}$ becomes

$$\bar{u} = \frac{\dot{V}}{\pi R^2} = - \frac{R^2}{8\mu} \frac{dp}{dx} \tag{4.53}$$

The maximum velocity, $u_{max}$, is derived by equating du/dr to zero at the location of maximum velocity, i.e.

$$\left[ \frac{du}{dr} \right]_{at\ u_{max}} = \left[ \frac{2r}{4\mu} \frac{\partial p}{\partial x} \right]_{at\ u_{max}} = 0 \tag{4.54}$$

This means that the maximum velocity occurs at the center line of the pipe and thus its value becomes

$$u_{max} = - \frac{R^2}{4\mu} \frac{\partial p}{\partial x} = 2\ \bar{u} \tag{4.55}$$

where $\bar{u}$ is the average velocity given by Eq. (4.53).

---

**EXAMPLE 4.4**

Starting from the basic principles find the velocity profile of steady incompressible flow between two concentric pipes of inner and outer radii of $r_1$ and $r_2$ respectively. The inner pipe is fixed while the outer pipe rotates with a constant angular velocity ω.

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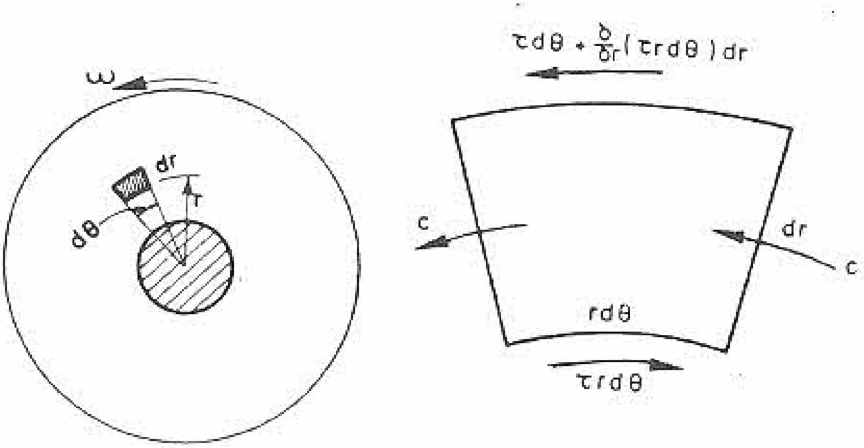
Problem Description



Data of the Problem

* two concentric pipes inner radius = $r_1$, outer radius = $r_2$
* outside pipe rotates with angular velocity $\omega$
* inner pipe is fixed

Requirements

* the velocity profile between the two pipes

Solution

The flow is unidirectional, axisymetric, and fully developed. Applying the momentum principle in the $\theta$ direction we get

$$\Sigma F_\theta = \dot{m}_o C_o - \dot{m}_i C_i$$

But since $\dot{m}_i = \dot{m}_o$ from conservation of mass and $C_i = C_o$ for fully developed flow, then

$$\frac{\partial}{\partial r} (\tau\ r\ d\ \theta) dr = 0$$

or

$$\frac{d}{dr} \left( \mu \frac{dC}{dr} r \right) = 0$$

by integration twice we get

$$C = a_1 \ln r + a_2$$

Boundary conditions

at $\quad r = r_1 \qquad C = 0$

$\quad r = r_2 \qquad C = \omega r_2$

Then

$$0 = a_1 \ln r_1 + a_2$$

$$\omega r_2 = a_1 \ln r_2 + a_2$$

i.e. $$a_1 = \frac{\omega r_2}{\ln \frac{r_2}{r_1}}$$

and

$$a_2 = -\frac{\omega r_2}{\ln\frac{r_2}{r_1}} \ln r_1$$

or

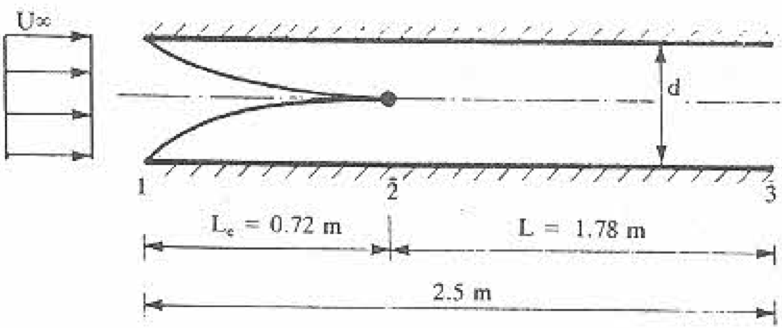
$$C = \frac{\omega r_2}{\ln\frac{r_2}{r_1}} (\ln r - \ln r_1) = \frac{\omega r_2}{\ln\frac{r_2}{r_1}} \ln \frac{r}{r_1}$$

$$C = \frac{\omega r_2}{\ln\frac{r_2}{r_1}} \ln \frac{r}{r_1}$$

### EXAMPLE 4.5

Air at 20°C and atmospheric pressure enters a 2.5 cm diameter pipe with a uniform velocity and a Reynolds number of 1000. Determine the decrease in pressure in going from the entrance to 250 cm downstream from the entrance. The entrance length
$L_e$ is given by $L_e/d = 0.0288\ Re$ , where Re is the Reynolds number.

Problem Description




Data of the Problem

* air at 20°C, atmospheric pressure
* Re = 1000
* d = 2.5cm
* $L_e/d$ = 0.0288Re

Requirements

* $P_1 - P_2$, as shown in Problem Description

Solution

From Appendix F, for air at 20°C have

$\rho = 1.21\ kg/m^3$

$\mu = 1.815 \times 10^{-5}\ Ns/m^2$

The value of the entrance length $L_e$ is

$$L_e = 0.0288 \times 1000 \times 0.025 = 0.72 \quad m$$

$$U_\infty = \frac{R_e \cdot \mu}{\rho d} = \frac{1000 \times 1.815 \times 10^{-5}}{1.21 \times 0.025} = 0.6 \quad m/s$$

The flow between sections 1 and 3 is divided into two regions: an entrance region (from 1 to 2) and a fully developed region (from 2 to 3). Since the flow inside the core is non-viscous the application of Bernoulli's equation on the stream line coinciding with the centerline of the pipe yields

$$P_1 - P_2 = \frac{1}{2}\rho \ (U_2^2 - U_\infty^2) \tag{I}$$

The velocity $U_2$ is the maximum velocity of the fully developed flow and is given by

$$U_2 = 2\ \bar{U} = 2\ U_\infty = 1.2 \quad m/s \tag{II}$$

where $\overline{U}$ is the average velocity in the pipe. Thus Eq. (I) becomes

$$P_1 - P_2 = \frac{1}{2} \times 1.21 \times [(1.2)^2 - (0.6)^2] = 0.653\ N/m^2$$

The flow between 2 and 3 is fully developed, thus

$$\bar{U} = U_\infty = \frac{-R^2}{8\mu} \quad \frac{\partial p}{\partial x}$$

or

$$\frac{\partial p}{\partial x} = -\frac{8\mu U_\infty}{R^2} = -\frac{8 \times 1.815 \times 10^{-5} \times 0.6}{(0.0125)^2}$$

$$= -0.558 \ (N/m^2)/m$$

By integration between 2 and 3 yields

$$P_2 - P_3 = 0.558 \times 1.78 = 0.-993 \quad N/m^2$$

Then

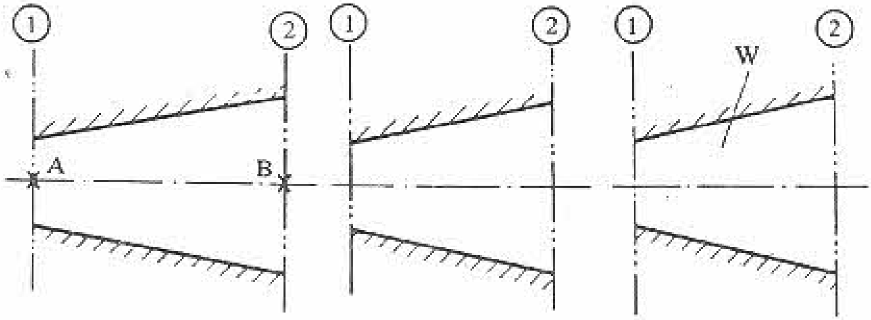
$$P_1 - P_3 = ( P_1 - P_2 ) + ( P_2 - P_3 )$$

$$= 0.653 + 0.992 = 1.645 \quad N/m^2$$

---

## 4 – 7 – Energy Degradation on Hydraulic Flow

Consider the pipe flow shown in Fig. (4.9). The flow is incompressible, steady, adiabatic and in a gravitational field with constant acceleration. If the fluid was non-viscous, Bernoulli's equation (Eq. 3.45 or 3.46) states that the high grade




a) ideal flow b) real flow c) real flow with work

*Fig. 4.9 Energy consideration in an adiabtic pipe flow*

energy per unit mass at any point on a streamline is constant. Thus applying Bernoull's equation between points (A) and (B) gives

$$\frac{P_A}{\rho} + g\ z_A + \frac{C_A^2}{2} = \frac{P_B}{\rho} + g\ z_B + \frac{C_B^2}{2} \tag{4.56}$$

Let us now consider the infinite stream tube connecting points A and B and having cross sectional areas $dA_1$ and $dA_2$ with points A and B in the center of these infinitesimal areas, respectively. Equation (4.56) can be rewritten as follows

$$\left(\frac{P}{\rho} + gZ + \frac{C^2}{2}\right)_A \cdot d\dot{m} = \left(\frac{P}{\rho} + gZ + \frac{C^2}{2}\right)_B \cdot d\dot{m}$$

By integration over the area of each section we conclude that

$$\int_{A_1} \frac{P}{\rho} \cdot \rho C dA_1 + \int_{A_1} gZ \, \rho C dA_1 + \int_{A_1} \frac{C^2}{2} \cdot \rho C dA_1$$

$$= \int_{A_2} \frac{P}{\rho} \cdot \rho C dA_2 + \int gZ \, \rho C dA_2 + \int \frac{C^2}{2} \, \rho C dA_2$$

where $A_1$ and $A_2$ are the flow areas at sections (1) and (2), respectively. The above equation can be integrated for constant pressure over the cross section and for no variation in Z for each section to give.

$$\frac{P_1}{\rho} + gZ_1 + \frac{1}{\bar{C}_1 A_1} \int_{A_1} \frac{C^3}{2} \, dA_1 = \frac{P_2}{\rho} + gZ_2 + \frac{1}{\bar{C}_2 A_2} \int_{A_2} \frac{C^3}{2} \, dA_2$$

or

$$\frac{P_1}{\rho} + gZ_1 + \alpha_1 \frac{\bar{C}_1^2}{2} = \frac{P_2}{\rho} + gZ_2 + \alpha_2 \frac{\bar{C}_2^2}{2} \tag{4.57a}$$

where $\overline{C}_1$ and $\overline{C}_2$ are the mean velocity at sections $A_1$ and $A_2$ respectively and $\alpha$ is the kinetic energy correction factor and defined as follows

$$\alpha_i = \frac{\int_{A_i} C_i^3 \, dA_i}{A \, \bar{C}_i^3} \tag{4.57b}$$

The value of $\alpha$ is 1 for one dimensional flow (i.e. C is constant over the cross section area) but otherwise $\alpha \neq 1$. If a real fluid is now considered, i.e. the effect of viscosity is introduced, Eq. (4.57) becomes incorrect. This is because some of the energy at any point upstream such as point 1 is degraded* by the time the fluid reaches a downstream point such as point 2 because of friction between the fluid and the wall and between fluid layers. A correction for Eq. (4.57) yields

$$\frac{P_1}{\rho} + gZ_1 + \alpha_1 \frac{\bar{C}_1^2}{2} = \frac{P_2}{\rho} + gZ_2 + \alpha_2 \frac{\bar{C}_2^2}{2} + E_D \tag{4.58}$$

where $E_D$ is the total degraded energy per unit mass because of friction, turbulence, and/or other effects. The above equation is called the energy equation and it is applied between two sections on the pipe. The velocities $\overline{C}_1$ and $\overline{C}_2$ are the average velocities on the pipe at sections 1 and 2 respectively. The energy equation can be further generalized by considering Fig. (4.9c) where work W per unit mass is added to or drawn from the flow between the two sections, thus

(*) Degradation of energy means transformation of energy from the form of work energy such as potential energy, kinetic energy, flow energy, mechanical or electric work,... etc to a non work form such as heat, internal energy, or chemical energy. Energy degradation per unit weight is usually known as head loss.

$$\frac{P_1}{\rho} + gZ_1 + \alpha_1 \frac{\bar{C}_1^2}{2} + W = \frac{P_2}{\rho} + gZ_2 + \alpha_2 \frac{\bar{C}_2^2}{2} + E_D \qquad (4.59)$$

The above equation states that for a control volume under steady flow conditions the sum of energy input per unit mass is equal to the sum of the energy output per unit mass. Equation (4.59) can only be applied to steady, incompressible and adiabatic flow.

The work W per unit mass is considered positive for a pump (work added to the flow) and negative in case of a turbine (work taken from the flow). The degraded energy in such flows, invariably transforms to internal energy.

Energy degradation in hydraulic conduits may be divided into two parts:

i) energy degradation due to friction in fully developed flow in straight constant cross section conduits usually known as major loss, $E_j$, and

ii) Energy degradation due to any reason other than the above usually known as minor loss $E_n$ (e.g. contractions, enlargements, entrances, elbows, valves, exits, .. etc.)

4 – 7 – 1 – **Major loss**

**a. Laminar flow**

Consider the fully developed laminar flow in straight pipe shown in Fig. (4.10) where there is no change in potential energy and a mean velocity $\bar{C}$ is assumed at every section. The energy equation can be written as follows

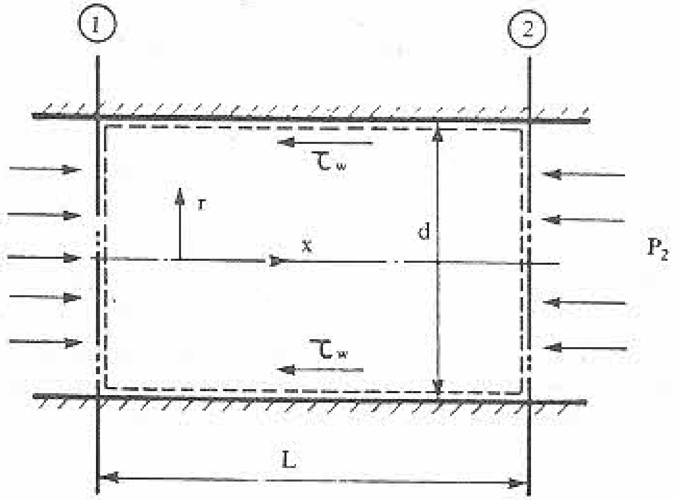



*Fig. 4.10 Fully developed laminar flow in a pipe*

$$\frac{P_1}{\rho} = \frac{P_2}{\rho} + E_j \tag{4.60}$$

Substituting from Eq. (4.55b) into Eq. (4.60) where $P_1 - P_2 = \Delta P$ and L in Fig. (4.2) is equal to x yields

$$E_j = \frac{8 \mu \dot{V} L}{\pi \rho R^4} \tag{4.61}$$

Major losses are commonly expressed in terms of the kinetic energy of the fluid ($\bar{C}^2/2$), thus $E_j$ may be expressed as

$$E_j = \frac{64 \mu L}{\rho d^2 \bar{C}} \cdot \frac{\bar{C}^2}{2} = f \cdot \frac{L}{d} \cdot \frac{\bar{C}^2}{2} \tag{4.62}$$

where

$$f = \frac{64 \mu}{\rho \bar{C} d} \tag{4.63a}$$

is known as friction factor. The quantity $\rho \bar{C} d/\mu$ is the Reynolds number, $Re_d$, based on the pipe diameter. The quantity L/d is a non-dimensional ratio representing the geometry of the pipe. Equation (4.63a) can now be expressed as follows

$$f = \frac{64}{Re} \tag{4.63b}$$

The relation between the friction factor and the skin friction coefficient $C_f$ defined by Eq. (4.20) is derived by applying the momentum principle in the x direction on the control volume of Fig. (4.8) which yields

$$(P_1 - P_2) \frac{\pi}{4} d^2 = \tau_w (\pi d L) \tag{4.64}$$

substituting for $\tau_w$ from Eq. (4.20) where $\bar{C}$ replaces U and for $(P_1 - P_2)$ from Eqs. (4.60) gives

$$E_j = 4 C_f \left(\frac{L}{d}\right) \frac{\bar{C}^2}{2} \tag{4.65}$$

By comparison with Eq. (4.62), we get

$$f = 4 C_f \tag{4.66}$$

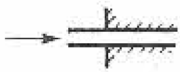
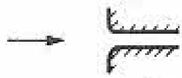
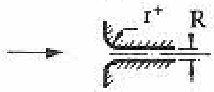
**b) Turbulent Flow**

For turbulent flow in pipes (i.e. $Re_d > 2300$) experiments showed that Eq. (4.62) is valid but with f being a function of both Reynolds number Re and the relative roughness $\varepsilon$, i.e.

$$f = f\ ( Re,\ \varepsilon\ ) \tag{4.67}$$

where $\varepsilon$ is defined as the average roughness divided by the diameter of the tube. Also, the value of the friction factor f is related to the skin friction coefficient $C_f$ by the same equation for laminar flow, i.e. Eq. (4.66), where $C_f$ is determined as follows [ Ref.: Kay and Nedderman, 1977 ].

Table 4.2 Loss coefficients for pipe entrances**

| ENTRANCE TYPE | *DIAGRAM | LOSS COEFFICIENT K* |
|---|---|---|
| Reentrant |  | 0.78 |
| Square edged |  | 0.34 |
| Slightly rounded |  | 0.2 – 0.25 |
| Well rounded |   | 0.04 |

* Based on $E_n = k\,(\bar{C}^2/2)$, where C is the mean velocity in the pipe.

\+ $r/R \simeq 0.35$

** Taken from: Fox, R.w. and A.T. McDonald, Introduction to Fluid Mechanics, John Wiley & Sons, 1978.

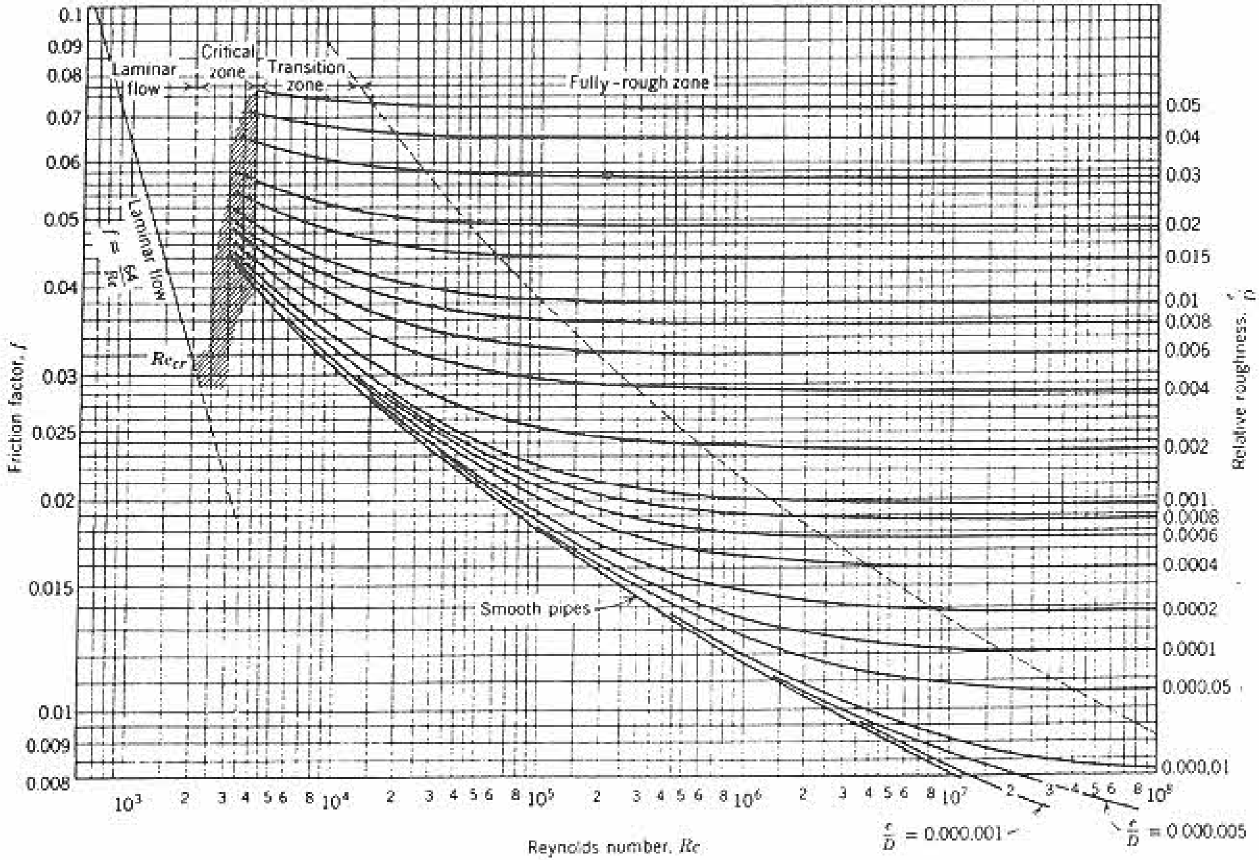



*Fig. 4.11 Friction factors for fully-developed flow in a circular pipes (Taken from: Fox, R.W. and A.T. McDonald, Introduction to Fluid Mechanics, John Wiley & Sons, 1978.)*

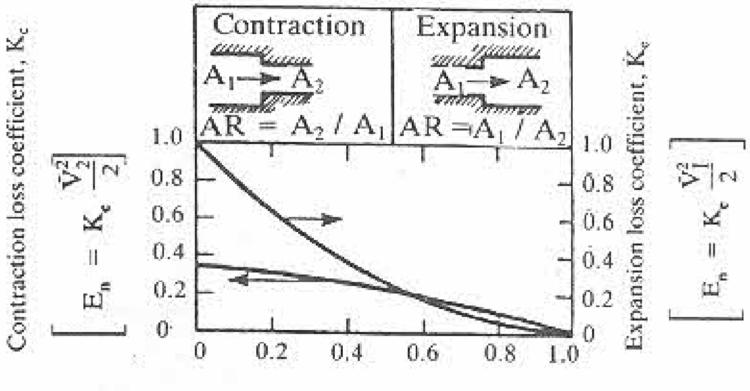



*Fig. 4.12 Loss coefficients for flow through sudden area changes (Taken from: Fox, R.W. and A.T. McDonald, Introduction to Fluid Mechanics, John Wiley & Sons, 1978).*

Table 4.3 Equivalent lengths in pipe diameters ($L_e/D$) for valves and Fittings**

| FITTING TYPE | DESCRIPTION | EQUIVALENT LENGTH $L_e/D$* |
|---|---|---|
| Globe Valve | Fully Open | 350 |
| Gate Valve | Fully Open | 13 |
| | $\frac{3}{4}$ Open | 35 |
| | $\frac{1}{2}$ Open | 160 |
| | $\frac{1}{4}$ open | 900 |
| Check Valve | | 50-100 |
| $90^{\circ}$ Std. Elbow | | 30 |
| $45^{\circ}$ Std. Elbow | | 16 |
| $90^{\circ}$ Elbow | Long Radius | 20 |
| $90^{\circ}$ Street Elbow | | 50 |
| $45^{\circ}$ Street Elbow | | 26 |
| Tee | Flow through run | 20 |
| | Flow through branch | 60 |
| Return bend | Close pattern | 50 |

* Based on $E_n = f \frac{L_e}{D} \frac{C^2}{2}$

** Taken from : Fox, R.W. and A.T. McDonald, Introduction to Fluid Mechanics, John Wiley & Sons, 1978.

$$C_f = 0.079\, Re^{-1/4} \quad (10^5 > Re > 2300, \text{ smooth pipe})$$

$$\frac{1}{\sqrt{C_f}} = 4 \log_{10} \left( \frac{Re}{\sqrt{C_f}} \right) - 0.4 \quad (R_e > 10^5, \text{ smooth pipe})$$

$$\frac{1}{\sqrt{C_f}} = 4 \log_{10} \left( \frac{d}{2e} \right) + 3.46 \quad (R_e > 10^5, \text{ very rough pipe}) \tag{4.68}$$

where e is the average roughness.

The value of the friction factor f defined by Eq. (4.67) was determined by L.F. Moody and this relation is shown in Fig. (4.11).

### 4 − 7 − 2 − **Minor losses**

Among these are the following: i) entrance loss, ii) bends and elbows loss, iii) exit loss, iv) valves and fittings loss, and v) enlargments and contractions losses. These losses are generally expressed in one of the following two forms, depending on the kind of loss

$$E_n = K \frac{\bar{C}^2}{2} \tag{4.69a}$$

or

$$E_n = f \left( \frac{L}{d} \right)_e \frac{\bar{C}^2}{2} \tag{4.69b}$$

where K is a constant known as loss coefficient, $(L/d)_e$ is an equivalent ratio of the length to diameter ratio and f is the same friction factor calculated for the major loss. Values of K or $(L/d)_e$ adopted for different kinds of minor losses are given in Tables (4.2) and (4.3) and in Fig. (4.12).

---

**EXAMPLE 4.6**

Determine the gauge pressure $P_1$ to produce a volume flow rate of $V = 0.5\ m^3/s$ for the configuration shown in the Problem Description. Consider the following properties for water: $\rho = 1000\ kg/m^3$ and $\mu = 0.001$ Pa.s.

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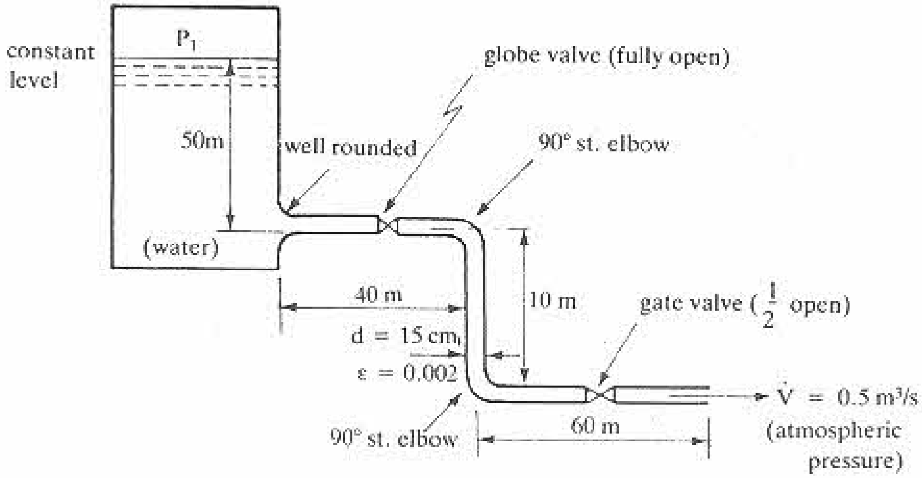
Problem Description




Data of the Problem

* as shown in the Problem Description
* $\rho_w = 1000\ kg/m^3$ , $\mu_w = 0.001$ pa.s

Requirements

* Gauge pressure, $P_{1,g}$

Solution

Select points (1) at the constant level and (2) at exit to atmosphere, then applying the energy equation between (1) and (2) gives

$$\frac{P_1}{\rho} + \frac{\bar{C}_1^2}{2} + gZ_1 = \frac{P_2}{\rho} + \frac{\bar{C}_2^2}{2} + gZ_2 + E_D$$

or

$$P_{1,g} = \rho g_1\ (Z_2 - Z_1) + \rho\frac{\bar{C}_2^2}{2} + \rho E_D \qquad \text{(I)}$$

where $P_2$ = atmospheric pressure and $\overline{C}_1 = 0$. The value of $\overline{C}_2$ is calculated from the volume flow rate as

$$\bar{C}_2 = \frac{\dot{V}}{\frac{\pi}{4}d^2} = \frac{0.5}{\frac{\pi}{4}(0.15)^2} = 28.3 \quad m/s$$

But,

$$E_D = E_j + E_n$$

$$= \left(f \frac{L}{d} \frac{C_2^2}{2}\right) + (E_{inlet} + 2E_{elbow} + E_{globe} + E_{gate})$$

$$= f \frac{L}{d} \frac{C_2^2}{2} + K_{in} \frac{C_2^2}{2} + 2f \left(\frac{L_e}{d}\right)_{elb} \frac{C_2^2}{2} + f \left(\frac{L_e}{d}\right)_{gobe} \frac{C_2^2}{2}$$

$$+ f \left(\frac{L_e}{d}\right)_{gat} \frac{C_2^2}{2}$$

$$= f \frac{C_2^2}{2} \left[ \frac{L}{d} + 2\left(\frac{L_e}{d}\right)_{elb} + \left(\frac{L_e}{d}\right)_{glob} + \left(\frac{L_e}{d}\right)_{gat} \right] + K_{inlet} \frac{C_2^2}{2}$$

$$= f \frac{C_2^2}{2} \left[ \frac{110}{0.15} + 2x30 + 350 + 160 \right] + 0.04 \frac{C_2^2}{2}$$

$$= 1303.3 \; f \frac{C_2^2}{2} + 0.04 \frac{C_2^2}{2} \qquad \text{(II)}$$

Also,

$$Re = \frac{\rho C d}{\mu} = \frac{10^3 x 28.3 x 0.15}{0.001} = 4.24 \times 10^6 > 2300$$

i.e. the flow is turbulent. Using Fig. (4.1) for $\epsilon = 0.002$ then $f = 0.0242$, i.e.

$$E_D = 1303.3 \times 0.0242 \times \frac{(28.3)^2}{2} + 0.04 \times \frac{(28.3)^2}{2}$$

$$= 12630.3 + 16.01 = 12646.31 \quad m^2/s^2$$

Therefore Eq. (I) becomes

$$P_{1,g} = -\rho g \, (Z_2 - Z_1) + \rho \frac{C_2^2}{2} + \rho E_D$$

$$= -10^3 \times 9.8 \times 60 + 10^3 \times \frac{(28.3)^2}{2} + 10^3 \times 12646.3$$

$$= -5.88 \times 10^5 + 4 \times 10^5 + 126.46 \times 10^5$$

$$= 124.58 \times 10^5 \; N/m^2 = 1.24 \times 10^7 \; N/m^2$$

**EXAMPLE 4.7**

Water flows from the pipe connected to the tank shown in the Problem Description. Assume a smooth pipe with well rounded entrance, determine the volume flow rate in the pipe ($\mu_w = 10^{-3}$ Pa.s).

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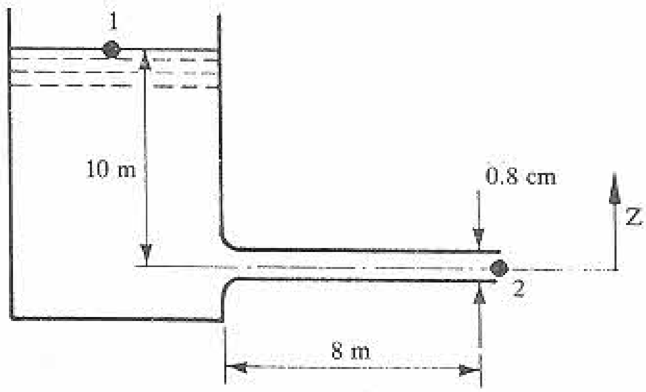
Problem Description




Data of the Problem

* water flowing from a constant level tank as shown in the Problem Description

Requirements

* volume flow rate from the tank

Solution

Apply the energy equation between points 1 and 2

$$\frac{P_1}{\rho} + gZ_1 + \frac{\bar{C}_1^2}{2} = \frac{P_2}{\rho} + gZ_2 + \frac{\bar{C}_2^2}{2} + E_D \qquad \text{(I)}$$

Since $P_1 = P_2 = P_{atm}$ and $\bar{C}_1 = 0$, then

$$\frac{\bar{C}_2^2}{2} + E_D = g\, Z_1 \qquad \text{(II)}$$

The total friction loss can be expressed as follows

$$E_D = E_j + E_n$$

$$E_D = f\left(\frac{L}{d}\right)\left(\frac{\bar{C}_2^2}{2}\right) + K_{ent}\left(\frac{\bar{C}_2^2}{2}\right) \qquad \text{(III)}$$

where $K_{ent}$ is loss coefficient for entrance. Equation (II) therefore becomes

$$\frac{\bar{C}_2^2}{2}\left(1 + f\,\frac{L}{d} + K_{ent}\right) = g\,Z_1$$

or

$$\bar{C}_2 = \sqrt{\frac{2g\,Z_1}{1+f\,\frac{L}{d} + K_{ent}}} \qquad \text{(IV)}$$

The value of $K_{ent}$ is determined from Table (4.2) as $K_{ent} = 0.04$. Equation (IV) is a non-linear algebraic equation in $\overline{C}$. The value of f depends on the value of $\overline{C}$. The procedure to solve Eq. (IV) is a trial and error one. To start the solution an initial guess is to be assumed for $\overline{C}$. A good estimate for $\overline{C}$ might be slightly less than in the pipe assuming non-viscous fow. Thus

$$\bar{C}_2 = \sqrt{2g\,(Z_1 - Z_2)} = \sqrt{2 \times 9.8 \times 10} = 14 \text{ m/s}$$

Then assume the initial guess for $\overline{C}_2$ to be 10m/s, thus

$$R_e = \frac{\rho\,\bar{C}_2\,d}{\mu} = \frac{10^3 \times 10 \times 0.008}{10^{-3}} = 8 \times 10^4$$

i.e. the flow is turbulent. Using Moody chart, for smooth pipe the value of f is 0.0175. The L.H.S. of Eq. (IV) becomes

$$\bar{C}_2 = \sqrt{\frac{2 \times 9.8 \times 10}{1 + 0.0175 \times \frac{8}{0.008} + 0.04}} = 3.25 \text{ m/s}$$

of course the value for $\overline{C}_2$ differs than the initial guess. A new trial will be done with the new estimate of $\overline{C}_2 = 3.25$. The process continues until the value of $\overline{C}_2$ calculated from Eq. (IV) becomes almost like the initial guess to a certain accuracy. The following tabulation summarizes this procedure.

| Trial No. | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\bar{C}_2$, guess | 10 | 3.25 | 2.717 | 2.702 |
| $R_e$ | $8\times10^4$ | $2.6\times10^4$ | $2.17\times10^4$ | $2.15\times10^4$ |
| f(from Moody chart) | 0.0175 | 0.0255 | 0.0258 | 0.0258 |
| $\bar{C}_2$, from Eq. (IV) | 3.25 | 2.717 | 2.702 | 2.702 |

i.e. the value of $\bar{C}_2$ is 2.702 m/s. Therefore,

$$\dot{V} = \bar{C}_2 \, A_2 = 2.702 \times \frac{\pi}{4} (0.008)^2$$
$$= 1.36 \times 10^{-4} \quad m^3/s$$
$$= 0.49 \quad m^3/hr$$

### 4 – 7 – 3 – Losses in non-circular ducts

If the duct carrying the flow has a non-circular cross section, Equation (4.62) can still be used to estimate approximately the major loss of energy per unit mass, $E_j$, with the same expression for f as given by Eq. (4.63) for laminar flow but after replacing the value of 64 by the constant B that depends on the geometry of the cross section of the duct.

Table 4.4 Friction factors for concentric annulus and rectangle laminar flow using the equation f = B/Re*

*Annulus* — $r_1$ and $r_2$ are inner and outer radii respectively

| $r_1/r_2$ | B |
|---|---|
| 0.0001 | 71.78 |
| 0.001 | 74.68 |
| 0.01 | 80.11 |
| 0.05 | 86.27 |
| 0.10 | 89.37 |
| 0.20 | 92.35 |
| 0.40 | 94.71 |
| 0.60 | 95.59 |
| 0.80 | 95.92 |
| 1.00 | 96.00 |

*Rectangle* — dimensions a × b

| a/b | B |
|---|---|
| 0 | 96.00 |
| 1/20 | 89.91 |
| 1/10 | 84.68 |
| 1/8 | 82.34 |
| 1/6 | 78.81 |
| 1/4 | 72.93 |
| 2/5 | 65.47 |
| 1/2 | 62.19 |
| 3/4 | 57.89 |
| 1 | 56.91 |

* Taken from: Essentials of Engineering Fluid Mechanics, by R.M. Olson, p. 288, Fourth Edition, Harper & Row, Publishers, New York, 1980.

Values of B for different geometries are shown in Table (4.4). However, in case of turbulent flow the value of f calculated from Eq. (4.67) for circular duct is possibly used for noncircular ducts. In either case of laminar or turbulent flow the hydraulic diameter $d_h$ of the non-circular duct replaces the diameter d in calculating $E_j$ from Eq. (4.62) or in calculating the value of Reynolds number. The hydraulic diameter is defined as follows

$$d_h = 4 \frac{\text{cross sectional area of flow}}{\text{wetted perimeter}} \tag{4.70}$$

The wetted perimeter is taken as the length of the wall of the duct that is in contact with the flow at any cross section. For example, for a circular duct of diameter d we can write

$$d_h = 4 \times \frac{\frac{\pi}{4} d^2}{\pi d} = d$$

while for a rectangular duct of dimension a × b we get

$$d_h = 4 \times \frac{a\ b}{2(a+b)} = \frac{2\ a\ b}{a + b}$$

When the flow takes place in the annulus between two concentric pipes with inner and outer diameters $d_1$ and $d_2$ respectively the hydraulic diameter becomes

$$d_h = 4 \times \frac{\frac{\pi}{4} (d_2^2 - d_1^2)}{\pi\ (d_2 + d_1)} = d_2 - d_1$$

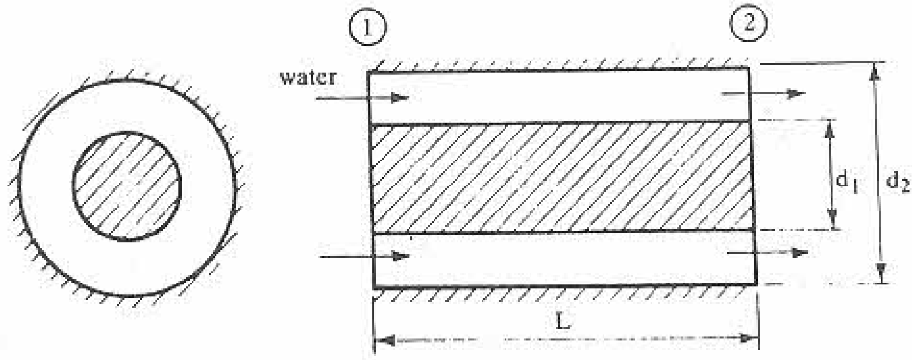
Unfortunately, minor losses are not calculated in a simple way (via using $d_h$) as the major loss. Experimental data for each geometery of the cross section of the non circular ducts should be consulted in order to get a reasonable estimate for the minor losses in these non-circular ducts.

**EXAMPLE 4.8**

Water flows in the annulus between two circular ducts of diameters 20 cm and 40 cm with an average velocity 5 m/s. Estimate the pressure drop per unit length of the duct (take $\rho = 1000\ kg/m^3$ and $\mu = 0.001$ Pa.s)

---

Problem Description




Data of the Problem

* water flowing in the annulus shown above
* $d_1 = 0.2$, $d_2 = 0.4$ m, L = 1 m
* $\rho = 1000$ kg/m$^3$, $\mu = 0.001$ Pa.s
* $\overline{C} = 5$ m/s

Requirement

* $\Delta P = P_1 - P_2$

Solution

Appling the energy equation between points (1) and (2) yields

$$\frac{P_1}{\rho} = \frac{P_2}{\rho} + E_D \tag{I}$$

Since only major loss encountered here, then $E_D = E_j$ where $E_j$ is given by

$$E_j = f \cdot \frac{L}{d_h} \cdot \frac{\bar{C}^2}{2} \tag{II}$$

The value of $d_h$ is given by Eq. (4.70) as follows

$$d_h = 4 \times \frac{\frac{\pi}{4}(d_2^2 - d_1^2)}{(d_2 + d_1)} = d_2 - d_1 = 0.4 - 0.2 = 0.2 \text{ m}$$

The Reynolds number thus becomes

$$R_e = \frac{\rho \; \bar{C} \; d_h}{\mu} = \frac{10^3 \times 5 \times 0.2}{0.001} = 10^6$$

i.e. the flow is turbulent

Assuming a smooth duct,then using Moody chart the value of f is 0.0115. Equations (I) and (II) are now combined to calculate $(P_1 - P_2)$ as follows

$$P_1 - P_2 = \rho \cdot f \cdot \frac{L}{d_h} \cdot \frac{\bar{C}^2}{2}$$

$$= 10^3 \times 0.0115 \times \frac{1}{0.2} \times \frac{25}{2} = 718.73 \quad Pa$$

---

### 4 – 7 – 4 – Energy and hydraulic gradients

If Eq. (4.59) is rewritten in the form

$$\frac{P_1}{\rho g} + z_1 + \frac{\bar{C}_1^2}{2g} + \frac{W}{g} = \frac{P_2}{\rho g} + z_2 + \frac{\bar{C}_2^2}{2g} + \frac{E_d}{g} \qquad (4.71)$$

where all terms now have the dimensions of length, the sum of the high grade energy terms will represent the total high grade energy head and the degraded energy term $E_d/g$ represents the hydraulic head loss usually known as the head loss. Equation (4.71) may be rewritten in the following form

$$\frac{P_1}{\rho g} + z_1 + \frac{\bar{C}_1^2}{2g} + H_p = \frac{P_2}{\rho g} + z_2 + \frac{\bar{C}_2^2}{2g} + H_L \qquad (4.72)$$

where $H_p = W/g$ is the head of a pump (positive) or a turbine (negative), and $H_L = E_d/g$ is the head loss.

When the total high grade energy head is plotted with location it gives what is known as the energy gradient. However, when the kinetic energy head $(\bar{C}^2/2g)$ is excluded from the total high grade energy head and the result is plotted with location it gives what is usually known as the hydraulic gradient.

Consider for example the flow system shown in Fig. (4.13),The system consists of constant level tank, pipe A of diameter $d_A$, pipe B of diameter $d_B$ four 45° elbows, a globe valve, and a pump. Applying Eq. (4.72) between locations (1) and (2) gives

$$\frac{P_1}{\rho g} + z_1 + \frac{\bar{C}_1^2}{2g} = \frac{P_2}{\rho g} + z_2 + \frac{\bar{C}_2^2}{2g} + H_{ent} \qquad (4.73)$$

where $H_{ent}$ is the entrance loss. In the above equation $\overline{C}_1 = 0$, and using gage pressure Eq. (4.73) becomes

$$z_1 = \frac{P_2}{\rho g} + z_2 + \frac{\overline{C}_2^2}{2g} + H_{ent}$$

i.e. the energy at location (1) is only that of the potential energy. This means that the fluid level in the tank represents the energy gradient line, and since $\overline{C}_1 = 0$, it also represents the hydraulic gradient line. At location (2) the total high grade energy head is decreased by an amount of $H_{ent}$ due to the entrance loss. At this location some of the potential energy is transformed into kinetic energy. The hydraulic gradient line at location (2) comes lower than the energy gradient line by $\overline{C}_2^2/2g$. Where $\overline{C}_2 = \overline{C}_A$ is the mean velocity in pipe A.

Similarly by applying Eq. (4.72) between every two successive points from point (2) to point (15) and carrying out the necessary simplifications we get.

$$\frac{P_2}{\rho g} = \frac{P_3}{\rho g} + H_{maj,2-3} \tag{4.74}$$

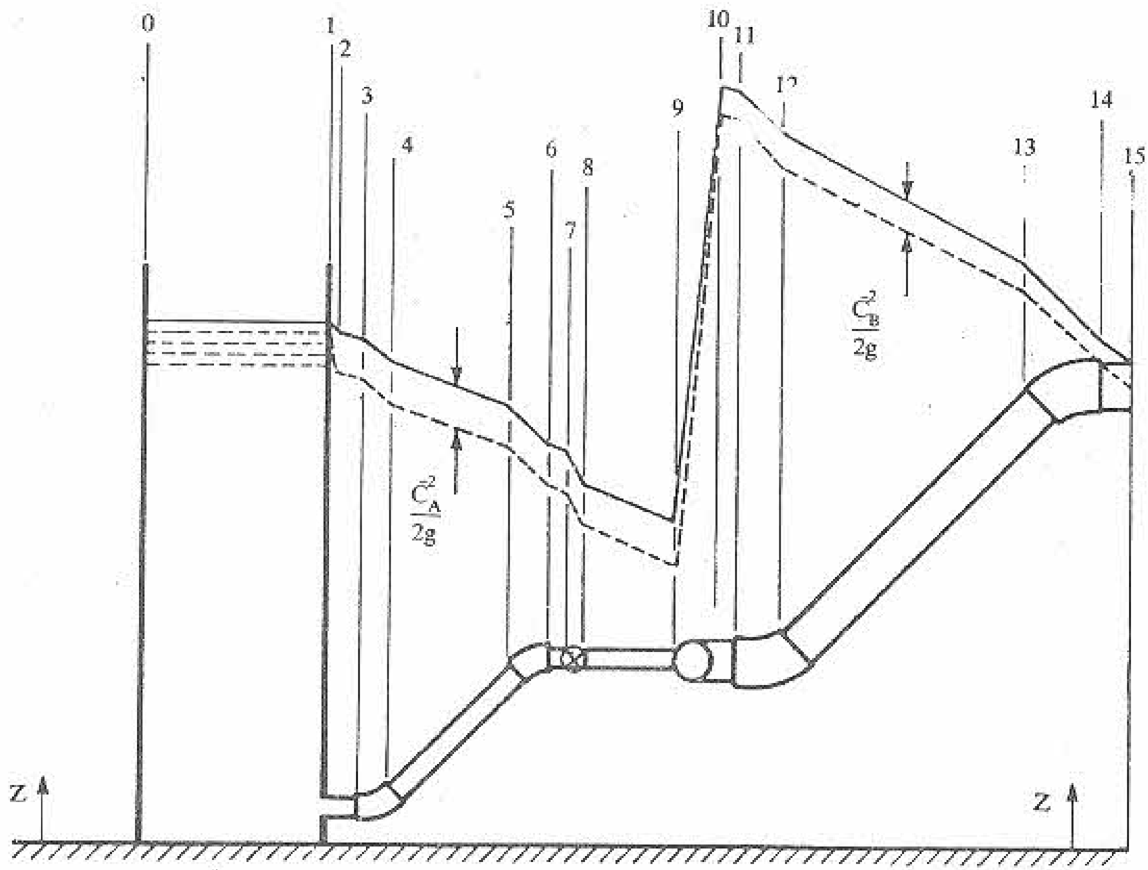
$$\frac{P_3}{\rho g} = \frac{P_4}{\rho g} + H_{elbow} \tag{4.75}$$

$$\frac{P_4}{\rho g} + z_4 = \frac{P_5}{\rho g} + z_5 + H_{maj,4-5} \tag{4.76}$$

$$\frac{P_5}{\rho g} = \frac{P_6}{\rho g} + z_6 + H_{elbow} \tag{4.77}$$

$$\frac{P_6}{\rho g} = \frac{P_7}{\rho g} + H_{maj,6-7} \tag{4.78}$$

$$\frac{P_7}{\rho g} = \frac{P_8}{\rho g} + H_{globe} \tag{4.79}$$




0-1 liquid level
1-2 entrance: square-edged
2-3 pipe A
3-4 45° elbow
4-5 pipe A
5-6 45° elbow
6-7 pipe A
7-8 globe valve
8-9 pipe A
9-10 pump
10-11 pipe B
11-12 45° elbow
12-13 pipe B
13-14 45° elbow
14-15 pipe B

*Fig. 4.13 Energy gradient line ( ——— ) and hydraulic gradient line ( – – – – )*

$$\frac{P_8}{\rho g} = \frac{P_9}{\rho g} + H_{maj,8-9} \tag{4.80}$$

$$\frac{P_9}{\rho g} + \frac{\bar{C}_A^2}{2g} + H_{pump} = \frac{P_{10}}{\rho g} + \frac{\bar{C}_B^2}{2g} \tag{4.81}$$

$$\frac{P_{10}}{\rho g} = \frac{P_{11}}{\rho g} + H_{maj,10-11} \tag{4.82}$$

$$\frac{P_{11}}{\rho g} = \frac{P_{12}}{\rho g} + H_{elbow} \tag{4.83}$$

$$\frac{P_{12}}{\rho g} + Z_{12} = \frac{P_{13}}{\rho g} + Z_{13} + H_{maj,12-13} \tag{4.84}$$

$$\frac{P_{13}}{\rho g} = \frac{P_{14}}{\rho g} + H_{elbow} \tag{4.85}$$

$$\frac{P_{14}}{\rho g} = H_{maj,14-15} \tag{4.86}$$

where

| | |
|---|---|
| $H_{maj,i-j}$ | is the major head loss in pipe A or B between points i and j |
| $H_{elbow}$ | is the head loss in the 45° elbow |
| $H_{globe}$ | is the head loss in the globe valve |
| $H_{pump}$ | is the head added by the pump |
| $\overline{C}_A$, $\overline{C}_B$ | are the mean velocities in pipes A and B respectively. |

## PROBLEMS ON CHAPTER FOUR

**Problems on Sections 4-1 to 4-4**

4.1. Standard air ( $\frac{\mu}{\rho}$ = 1.46 x $10^{-5}$ $m^2/s$) flows along a flat plate at a velocity $U_\infty = 7.5$ m/s. Estimate the boundary layer thickness at 15 cm from the leading edge.

4.2. Calculate the boundary layer thickness of a flow of air over a flat plate at distance of 0.5 m from the leading edge. The air has a uniform velocity equals to 2 m/s. Note that: $\mu_{air} = 18.76 \times 10^{-6}$ Pa.s, $\rho_{air}$ = 1.23 kg/m$^3$, and $Re_{critical} = 3.2 \times 10^5$.

4.3. Water at 20°C flows over a flat plate aligned with the flow. If the uniform velocity of the water outside the boundary layer is 10 m/s, determine the distance along the plate at which transition occurs.

4.4. Repeat the previous problem for

a) Benzene at 20°C

b) Air at 20°C

4.5. An aeroplane is moving at a velocity of 100 km/hr for take-off. The wing of the aeroplane has a dimension of 2 m long and 15 m width. Assuming the wing to be a flat plate and that surrounding air is at 20°C and atmospheric pressure, calculate the boundary layer thickness at the trailing edge of the wing.

4.6. Find the relation between the ratio $\delta/x$ and the local Reynolds number along a flat plate for the following velocity profile in a laminar boundary layer over a plate

$$\frac{u}{U_\infty} = \frac{5}{3}\left(\frac{y}{\delta}\right) - \frac{2}{3}\left(\frac{y}{\delta}\right)^3$$

where y is the diatance from the plate and x is the distance along the plate.

4.7. Air at 20°C and atmospheric pressure flows over a flat plate at velocity $U_\infty = 8$ m/s. Calculate the boundary layer thickness at 2, 10, and 20cm from the leading edge.

4.8. The velocity profile in the boundary layer of a two dimensional laminar flow along a flat plate is assumed as follows

$$\frac{u}{U_\infty} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

Find the expressions for

a) the growth of the boundary layer thickness as a function of x.

b) the displacement thickness as a function of x.

**Problems on Section 4-5**

4.9. Calculate the drag force on a sphere of diameter 2 cm. The sphere is placed in a stream of air of uniform velocity of 50 m/s. The air has a density and a viscosity of 1.23 kg/m$^3$ and $10^{-5}$ Pa.s, respectively. You may take the drag coefficient as 1.2 based on the projected area of the sphere in direction of flow.

4.10. A 6 mm diameter cylinder of 10 mm length is used as a pitot cylinder to measure the velocity distribution in a water tunnel. If the velocity at a given test section is very nearly uniform and equal to 15 m/s, estimate the drag force on the cylinder.

4.11. Water at 20°C flows at a uniform velocity over a 15 cm sphere. What is the velocity of the water that gives a drag of 10 N on the sphere.

4.12. In the previous problem if the sphere is replaced by a cylinder of 15 cm diameter and 20 cm length calculate the drag force on the cylinder.

4.13. Calculate the number of parachutes, each 30 m diameter, that should be used to drop a load of 50 kN at a terminal speed of 10 m/s through air at 20°C and 100 kPa abs. Take $C_D$ of the parachute as 1.2, based on the projected area normal to motion.

4.14. A body of a mass about 200 kg is dropped from an aeroplane after attaching it to a disc of diameter d to limit the terminal velocity to 12 m/s. Assuming the disc is made of light material of negligible weight as compared to the body, calculate the diameter of the disc if air is assumed to be at 20°C and $10^5$ Pa abs.

4.15. Compare the drag forces for the following objects when placed in a flow of air at 20°C and 100 kPa abs. of velocity 100 km/hr.

a) hollow hemisphere, d = 1 m

b) solid hemisphere, d = 1 m

c) circular cylinder normal to flow, d = 1 m, L = 0.5 m.

4.16. Calculate the aerodynamic force (drag force) of wind blowing on a chimeny of 1.5 m diameter and 10 m height. Take the wind speed as 30 km/hr and the air conditions as 30°C and 100 kPa abs.

4.17. Repeat the previous problem for a similar chimeny with squar cross section of 1.5 m side.

**Problems on Section 4-6**

4.18. Find the velocity distribution for a flow between two vertical plates where one of the plates is fixed and the other is moving with a velocity $U_o$. You may assume laminar, incompressible, steady, fully developed flow.

4.19. Find the velocity distribution for a flow in the annulus between two concentric pipes with their centerline making an angle $\theta$ with the horizontal plane. The inner pipe moves with velocity $U_o$ while the center pipe is fixed. Assume laminar, incompressible, steady, fully developed flow.

4.20. Consider fully-developed laminar flow in the annulus between two concentric pipes. The inner pipe is stationary, and the outer pipe moves in the x-direction with velocity $V_0$. Assume the axial pressure gradient to be zero ($dp/dx = 0$), obtain the velocity distribution in the annulus.

4.21. A sealed journal bearing is formed from concentric cylinders. The inner and outer radii are 2.5 and 2.6 cm, respectively. The journal length is 10 cm and it turns at 240 rpm. The gap is filled with oil in laminar motion. The velocity profile is linear across the gap. The torque needed to run the journal is 2 cm-kgf. Calculate the viscosity of the oil. Will the torque increase or decrease with time? why?

**Problems on Section 4-7**

4.22. Water ($\nu = 1.14 \times 10^{-6} m^2/s$) flows in a 15-cm pipe at a rate of 90 L/s. The total head drops 3.8 m between two sections 30 m apart along the pipe.

a) What is the friction factor?

b) What is the relative roughness of the pipe?

4.23. Water ($\nu = 1.14 \times 10^{-6} m^2/s$) flows through a 20-cm pipe which enlarges suddenly to 40 cm in diameter. A differential manometer containing mercury shows a deflection of 12 cm when connected across the enlargement. What is the flow rate?

4.24. The average velocity of a jet of water issuing from a round hole in the side of an open tank is 9.7 m/s. The hole is 4.9 m below the free surface of the water in the tank. What is the head loss due to viscous effects?

4.25. Calculate the pressure drop for a flow of crude oil (specific gravity $s = 0.87$ and kinematic viscosity is $4.6 \times 10^{-6} m^2/s$) in a cast iron pipe of diameter of 30 cm and length of 3 km. The average velocity of the oil may be considered as 2.5 m/s.

4.26. A 15-cm pipe is joined to a 30-cm pipe by a reducing flange. For water flowing at a rate of 115 L/s, what is the head loss when the flow is from the smaller to the larger pipe?

4.27. Water flows through a 20 cm diameter pipe of relative roughness 0.003 with volumetric rate of 0.08 $m^3/s$. Determine the pressure drop over 8 meter length of the pipe. The flow is fully developed and $\mu_w = 10.046 \times 10^{-4}$ Pa.s.

4.28. A pipe line of 2 km length and diameter 15 cm is carrying drinking water at a rate of 20 kg/s. Calculate the power loss in the water due to friction. Consider major losses in the pipe and minor losses due to 100 gate valves and 50 globe valves, where all valves are fully opened. Note that $\mu_w = 10.046 \times 10^{-4} Ns/m^2$.

4.29. In Fig. (4.14) calculate the mass flow rate of water flowing from the exit of the pipe ($\mu = 4.7 \times 10^{-4}$ Pa.s.).

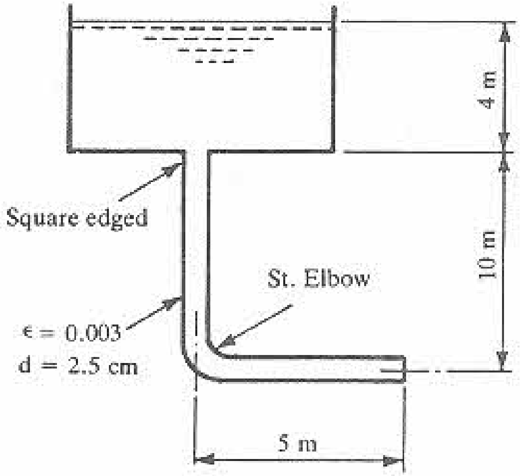



*Fig. 4.14*

4.30. Calculate the difference in pressure between the two plenum chambers about 12 m apart. The two chambers are connected by a smooth pipe of diameter 2.5 cm where air at 25°C flows from one chamber to the other at a velocity of 30 m/s. Neglect minor losses.

4.31. In Fig. (4.15) calculate the pressure $P_1$ of the oil in order to give an exit average velocity $\overline{C}_2 = 5.5$ m/s.

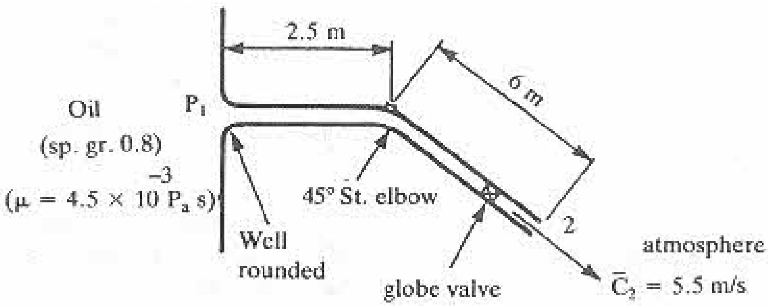



*Fig. 4.15*

4.32. What is the level, h, that must be maintained in the reservoir to produce a volumetric flow rate of 0.027 $m^3/s$. The inside diameter of the smooth pipe is 7.6 cm and the pipe length is 90 m. The loss coefficient k for the inlet is 0.5. The water discharges to the atmosphere.

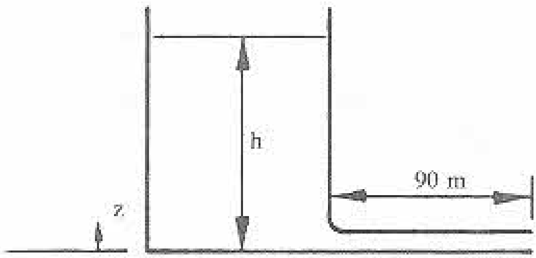



*Fig. 4.16*

**General Problems on Chapter 4**

4.33. The velocity distribution of air (at 27°C) in a turbulent boundary layer over a flat plate is given by

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7}$$

where U is the velocity of the free stream (=50 m/s) and $\delta$ is the boundary layer thickness. Calculate the displacement thickness at a location about 1.5m from the leading edge of the plate.

4.35. A compressed air drill require an air supply of 0.23 kg/s at pressure of $6.33 \times 10^5 N/m^2$ gauge at the drill. The hose from the air compressor to the drill has 3.75 cm inside diameter. The maximum compressor dischage pressure is $6.67 \times 10^5 N/m^2$ gauge. Neglect changes in density and any effects due to hose curvature. Air leaves the compressor at 38°C. Calculate the longest hose that may be used ($\mu_{air}$ at 38°C equals $18.76 \times 10^{-6}$ Pa.s, $R_{air} = 0.287$kJ/kg K).

4.36. A fire protection system is supplied from a water tower 24 m tall. The longest pipe in the system is 183 m, and is made of cast iron about 20 years old. The pipe diameter is 10 cm. Determine the maximum rate of flow through this pipe. $\mu_w = 10.046 \times 10^{-4}$ Pa.s.

4.37. Consider fully developed laminar flow between two plates where the upper plate is moving with velocity $U_1$ and the lower plate is moving with velocity $U_2$. Assuming zero pressure gradient along the channel, find the velocity distribution at any cross section and the average velocity.

4.38. In Fig. (4.17) the flow rate is 0.4 $m^3/s$ and the discharge is to atmospheric pressure. Determine the gauge pressure, $P_1$, to produce the flow.

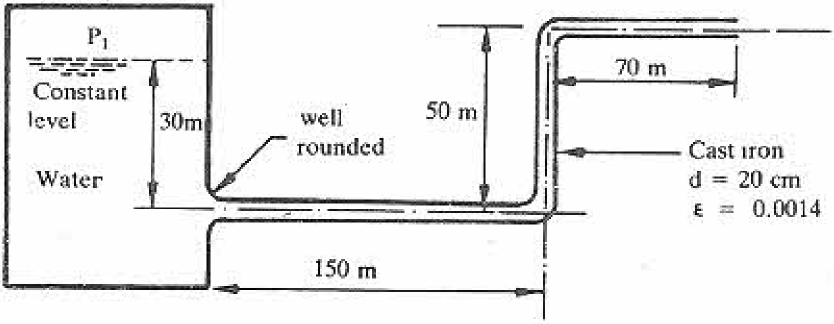



*Fig. 4.17*

4.39. In Fig. (4.18) determine the gauge pressure $P_1$ to produce the given flow rate ($\dot{V} = 0.5$ $m^3/s$). For water take $\mu_w = 0.001$ Pa.s

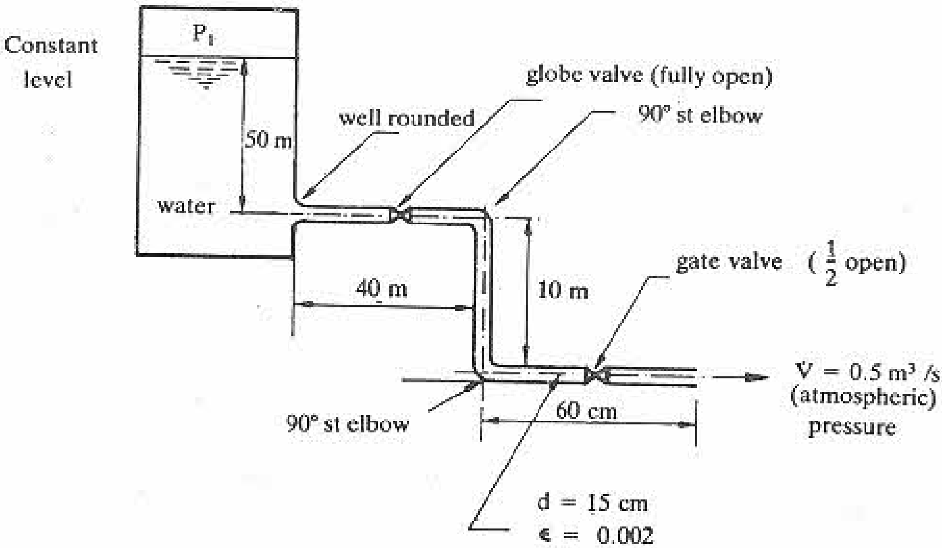



*Fig. 4.18*

4.40. Determine the shaft work of the pump needed to lift the water shown in Fig. (4.19) at a rate of 0.5 $m^3/min$. All pipes are 10 cm in diameter and they can be assumed smooth.

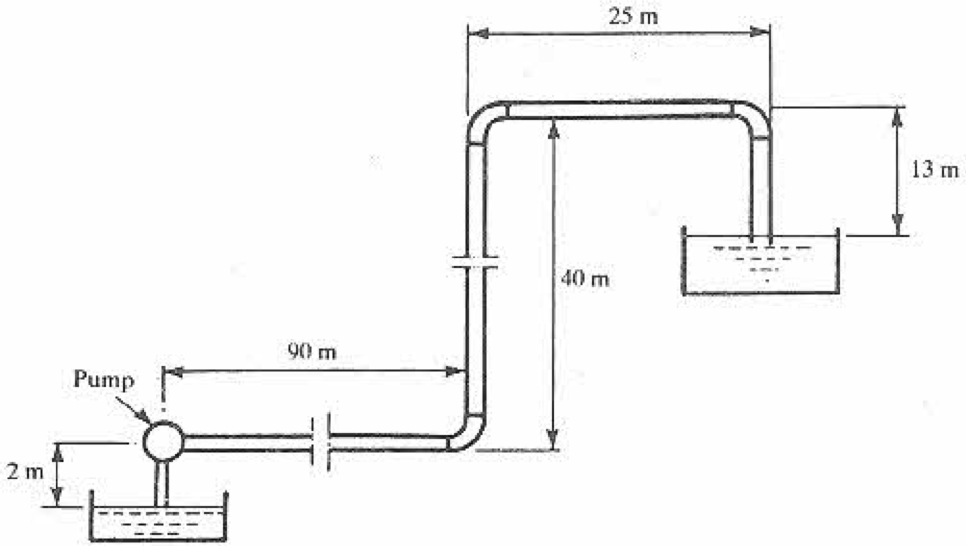



*Fig. 4.19*

4.41. In Fig. (4.20) determine the maximum total length, $L = L_1 + L_2 + L_3$, that gives the shown volume flow rate. For water take $\mu_w = 0.001$ Pa.s.

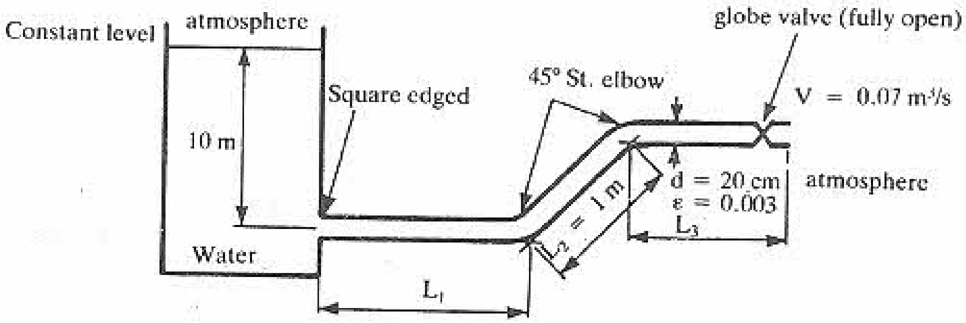



*Fig. 4.20*

CHAPTER FIVE

# INTRODUCTION TO COMPRESSIBLE FLOW

## 5 – 1 – Introduction

In the following we will introduce some of the basic relations governing the behaviour of a perfect compressible fluid. This analysis may be considered a continuation of chapter three where the basic relations governing the behaviour of non-viscous flows were discussed. The behaviour of actual compressible fluids is discussed in other texts.

Liquids may be considered incompressible for a good range of pressure variations. Compressibility in fluid flow exhibits its effects mainly with gases. This is usually demonstrated by changes in the density and temperature of gases undergoing compression or expansion processes. For this reason the behaviour of gases receives more analytical consideration in thermodynamic texts. Among these gases is the perfect gas. A perfect gas, beside being non-viscous, is assumed to possess other qualities-such as infinite conductivity, perfect emissivity and absorbtivity.. etc-that makes it a reversible medium. A reversible medium is that which does not degrade the availability of energy which it interacts with. Although a reversible medium is practically non-existent,there seems to be in existence some processes that occur almost in a perfect manner (that is a reversible manner). The travelling of sound in gases is one of these processes. Stagnation of a flowing fluid is another one.

A general reversible process in which the three fundamental properties; pressure p, volume V and temperature T of a gas change is known as a "reversible polytropic process" (see appendix C). Such process involves heat transfer between the gas and its surroundings. If the process is reversible but adiabatic (that is no heat transfer between the gas and its surroundings), then it is known as an "isentropic process". The following is a summary of the fundamental properties relations for a perfect gas undergoing a reversible polytropic process as deduced in appendix

$$PV^n = K_i \quad , \tag{5.1}$$

$$P\rho^{-n} = K_2 \qquad (5.2)$$

$$Tv^{n-1} = K_3 \qquad (5.3)$$

$$T\rho^{1-n} = K_4 \qquad (5.4)$$

$$TP^{\frac{1-n}{n}} = K_5 \qquad (5.5)$$

Also, the following is a summary of the fundamental properties relations for a perfect gas undergoing an isentropic process as deduced in appendix:

$$Pv^{\gamma} = K_6 \qquad (5.6)$$

$$P\rho^{-\gamma} = K_7 \qquad (5.7)$$

$$Tv^{\gamma-1} = K_7 \qquad (5.8)$$

$$T\rho^{1-\gamma} = K_9 \qquad (5.9)$$

$$TP^{\frac{1-\gamma}{\gamma}} = K_{10} \qquad (5.10)$$

A perfect gas undergoing either of the above processes would still be governed by the equation of state for gases, that is;

$$Pv = RT, \text{ or } P = \rho RT \qquad (5.11)$$

where:

- P is the pressure of the gas
- v is the specific volume of the gas
- Kn is a constant
- $n = 1 \rightarrow 10$
- R is the specific gas constant
- T is the absolute temperature of the gas
- n is the polytropic index for compression or expansion of the gas
- $\gamma$ is the isentropic index for compression or expansion of the gas, given as $\gamma = (C_P/C_v)$